# Napier and the Computation of Logarithms

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## Introduction

Some years ago it was claimed, in a widely read journal, that historical accounts [on Napier's Logarithms] *are either sketchy or inaccurate or both.*[1] True; but the situation has not really improved since then, especially considering the efforts of the claimant *to explain Napier's discovery* . I will concentrate here on one of the most neglected aspects of Napier's work on logarithms, namely on the numerical treatment of their construction. I have already done this for a smaller audience[2], by delivering a more or less detailed – albeit technical – sketch. The problem with historical numerics is that it usually has to involve many calculations and re-computations. This is especially (and trivially) true for Napier and his *Construction of the Wonderful Canon of Logarithms*[3], but might prove to be rather dull reading for those not especially interested in numerical methods. Instead I will try to make the present readers acquainted with some of Napier's ingenious numerical ideas. I shall try not to elaborate every detail, but will outline the most important lines of thought. I assume the reader to be acquainted with part of the literature on Napier's logarithms, but will state briefly the necessary facts and prerequisites. Some mathematics will also be needed, of course.

All Napierian ideas, theorems, and proofs will be presented here in modern terminology (but with the precautions necessary to avoid anachronisms). Napier's presentation of the *Constructio* is written in a deductive and laconic style that does not give any hints as to how his ideas evolved from each other. Any reasons why certain things have to be defined, introduced, or done in precisely the way they are defined, introduced, or done, have been carefully suppressed by Napier. This was the academic style of the time, and in this way Napier was able to produce one of the most influential and most dense mathematical texts on only 31 *octavo* pages of print.

## Napier's Logarithmic Function

**The function $SIN$.** It is well known today that Napier set out to construct a table of *logarithms of sines* for angles $\alpha$ in the first quadrant. The sine of an angle was given in Napier's day as the length of the leg opposite to $\alpha$ of a right triangle with hypotenuse $h$, and therefore depended on the choice of $h$. The larger $h$, the larger the sine and therefore the more accurate – because its length was given as a natural number (i.e. without decimals). Napier made use of a table of sines with $h = 10^7$. Let $SIN$ denote the sine belonging to a fixed hypotenuse of $10^7$; we then have:

$$\begin{aligned} \mathrm{SIN}(90^\circ) &= 10000000 \\ \mathrm{SIN}(60^\circ) &= 8660254 \\ \mathrm{SIN}(45^\circ) &= 7071068 \\ \mathrm{SIN}(30^\circ) &= 5000000 \end{aligned}$$

Thus, modern sines are obtained simply by dividing by $10^7$. Let Napier's logarithm be denoted by LN, to make its difference to the modern $LN$ (logarithmus naturalis) clear. Then Napier's aim can be stated as follows: construction of a table of $\mathrm{LN}(\mathrm{SIN}(\alpha))$ from $\alpha = 0°0'$ to $90°0'$ in steps of $1'$. This table will therefore have 5401 entries.

**The function LN.** Napier's logarithm LN is linked to our LN in the following way[4]:

$$\mathrm{LN}(x) = h \cdot (\mathrm{LN}(h) - \mathrm{LN}(x)) = h \cdot \mathrm{LN}\frac{h}{x}$$

Thus

$$\begin{aligned} \mathrm{LN}(\mathrm{SIN}(90^\circ)) &= \mathrm{LN}(10000000) &= 0 \\ \mathrm{LN}(\mathrm{SIN}(60^\circ)) &= \mathrm{LN}(8660254) &= 1438410 \\ \mathrm{LN}(\mathrm{SIN}(45^\circ)) &= \mathrm{LN}(7071068) &= 3465736 \\ \mathrm{LN}(\mathrm{SIN}(30^\circ)) &= \mathrm{LN}(5000000) &= 6931472 \end{aligned}$$

It is easy to see either immediately from the expression given above or from these examples that the sequence of figures in $\mathrm{LN}(\mathrm{SIN}(\alpha))$ is identical with the sequence of figures in $\mathrm{LN}(sin(\alpha))$; their respective values differ only in sign and in the position of the decimal point[5]. Napier's definition leads to $\mathrm{LN}(a \cdot b) = \mathrm{LN}(a) + \mathrm{LN}(b) - h \cdot \mathrm{LN}(h)$. Since he has $h = 10^7$, the modern logarithmic functional equation of type $log(a \cdot b) = log(a) + log(b)$ obviously does not hold. Apart from the fact that such a kind of functional equation was not what Napier looked for, this also was of no importance to him, because his logarithm LN already had all the properties Napier needed: for if $a \cdot b = c \cdot d$, then obviously $\mathrm{LN}(a \cdot b) = \mathrm{LN}(c \cdot d)$, but also $\mathrm{LN}(a) + \mathrm{LN}(b) - h \cdot \mathrm{LN}(h) = (c) + \mathrm{LN}(d) - h \cdot \mathrm{LN}(h)$. Now $-h \cdot \mathrm{LN}(h)$ can be cancelled, and we have $\mathrm{LN}(a) + \mathrm{LN}(b) = \mathrm{LN}(c) + \mathrm{LN}(d)$, which is easily seen to be able to replace our modern functional equation (see below).

[1] Ayoub (1993) 351.

[2] Fischer (1997)

[3] Napier (1619), Napier (1620), Napier (1889/1966).

[4] For details, the reader may consult one of the following books or articles: Napier (1889/1966), Glaisher (1911), Knott (1915), Naux (1966/71), Struik (1969, 1986), Ayoub (1993, 1994).

[5] By the way, the decimal point is Napier's invention too. – I will refrain from taking up the discussion whether Napier's logarithms are to the base $e$ or $e^{-1}$, whether natural logarithms are correctly Napierian logarithms according to some authors, and other questions of this kind. Let me add two remarks: 1. Scaling and changing the sign do not really create subtantially new numbers; 2. On the other hand, the *uncontrolled* use of anachronistic terminology – as *base* would be here – should be avoided.

It was only later that Napier had the idea of *further simplifying the use* of logarithms by a redefinition, which will not be discussed here, but which led to new logarithms – very similar to the decimal ones of our own contemporary tables. Napier's unexpected death left Henry[6] Briggs with the task of carrying this out.

**Properties of LN.** Napier proves the properties of LN geometrically. This was *state of the art* in his time; remember that many theorems later considered to belong to calculus (calculus itself not yet available in Napier's time) had already been proved in this way by Cavalieri, Descartes, Fermat, and others. Eudoxos of Knidos and Archimedes, however, were among the first to proceed in this way.

Here are the most important properties of LN (the § numbers are those of the *Constructio*):

§ 27: $\mathrm{LN}(h) = 0$

§ 29: $h - x < \mathrm{LN}(x) < \dfrac{h(h-x)}{x} \quad [0 < x < h]$

§ 34: $\mathrm{LN}(x)\mathrm{LN}(h) = \mathrm{LN}(x)$

§ 35: $\mathrm{LN}(b) + [\mathrm{LN}(a)\mathrm{LN}(b)] = \mathrm{LN}(a) \quad [a < b]$,
$\mathrm{LN}(a)[\mathrm{LN}(a)\mathrm{LN}(b)] = \mathrm{LN}(b) \quad [a < b]$

§ 36: $\mathrm{LN}(a)\mathrm{LN}(b) = \mathrm{LN}(u)\mathrm{LN}(v)$
if $a : b = u : v \quad [a < b]$

§ 38: $\mathrm{LN}(a) + \mathrm{LN}(v) = \mathrm{LN}(b) + \mathrm{LN}(u)$,
if $a : b = u : v$

§ 40: $\dfrac{h(b-a)}{b} < \mathrm{LN}(a) - \mathrm{LN}(b) < \dfrac{h(b-a)}{a}$
$[a < b]$

**Remarks.** The formulas given here correspond to Napier's *use,* but they should be taken with some care: Napier has no symbols for ordering at his disposal; neither was it customary in his time to distinguish "$<$" from "$\leq$" (which we today would prefer to see in almost all of the formulas); variable quantities may not assume negative values. Amendments in square brackets are conditions that he does not explicitly mention.

LN is monotonously decreasing from $\mathrm{LN}(0) = \infty$ to $\mathrm{LN}(h) = 0$ (the latter by § 27). Its range of arguments is *per definitionem* the closed interval $[0, h] = [0, 10^7]$, because its arguments are SINes for angles $\alpha$ between 0° and 90°. § 34 is obvious from § 27. § 35 expressly says that LN is a decreasing function, because the difference in brackets is always positive if $a < b$; the double statement is needed here, because the difference – a variable quantity – may not assume negative values (see above). The modern functional equation is replaced by §§ 36 and 38 – this being even an advantage for Napier's contemporaries, because they usually dealt with proportions. Most important is the inequality of § 40 (see below); § 29 is merely a special case of § 40 (let $a \to x, \quad b \to h$).

All of the inequalities may be derived by looking at Taylor's expansion of LN for suitable argument values and by breaking off after the linear term. But this procedure was not available in Napier's time (instead he used the geometrical methods mentioned above), and I will not pursue this interpretation further, interesting as it might be.

## Napier's ingenious approach

**The basic ideas.** It is easy *for us* to state in modern terminology what Napier's ingenious invention and construction is all about: *continuity* and *interpolation.* Both terms probably did not have any meaning for him, and surely not our modern meanings, but nonetheless they perfectly describe what he did in conceiving of LN.

First, all of his predecessors (and also his contemporary, Jost Bürgi) only saw that if an arithmetic sequence $y_n,\ y_n = y_0 + nd,\ n = 0, 1, 2, \ldots$, is linked to a geometric sequence $x_n,\ x_n = x_0 q^n,\ \ n = 0, 1, 2, \ldots$, in such a way that $x_n$ corresponds to $y_n$ and vice versa, then addition in the arithmetic sequence *somehow* corresponds to multiplication in the geometric sequence and *vice versa.* If "0" occurs in the arithmetic sequence and "1" in the geometric sequence, and if these two values correspond to each other, the resulting correspondence between addition and multiplication is simplest. In other cases the word *somehow* applies, but the correspondence is simple nonetheless. The problem with any geometric sequence $x_n$, however, is that sooner or later the distance between two consecutive terms $x_{n+1}$ and $x_n$ becomes too large or too small to be of any use in real calculations (this depends on $x_0$, $q$ and $n$, of course). Thus the question of *interpolation* should pose itself soon, but how could it possibly have been achieved? Bürgi decided in favor of a $q$ near to 1 – more precisely $q = 1.0001$ – and on what we today would call linear interpolation, but this is not reliable everywhere. So it seems tempting to stick to the discrete approach, because the final aim obviously should be a table, also having discrete entries. On the other hand, there was no satisfactory solution to the interpolation problem then. Napier's LN, however, is *continuous* in its whole range of definition – with the exception of $x = 0$, where $\mathrm{LN}(x)$ becomes infinite – so the relationship between $x$ and $y = \mathrm{LN}(x)$ exists everywhere, not only at discrete points (as in sequences). The interpolation problem seems to transform itself into a merely computational problem.

Next, Napier embeds the discrete approach into his continuous one (see below), and therefore the interpolation problem continues to exist for him, too. This problem, however, is now solved with the sole help of inequality § 40. Since Napier – by the *continuity* of his LN – is free to choose the $a$'s and $b$'s of § 40 as close together as he wishes (or needs, as we soon will see), interpolation now can be made arbitrarily accurate. In this way, the detour that he seems to take, by inventing the admit-

[6]Not *John,* as Ayoub erroneously writes two times: (Ayoub [1993] 363).

tedly more complicated continuous relationship between $x$ and $y$, pays off immediately with the satisfactory and complete solution of the interpolation problem.

## Before the Construction

**Initial range.** Napier only has at his disposal the derived properties of LN and the single value 0 of LN at $x = h$. This means that from $\mathrm{LN}(h) = 0$ and these properties *all other values* have to be computed. Computation here means to determine numerically the value of the function LN at (potentially) any argument $x$. Therefore an initial interval or range of computation has to be chosen. Napier could have chosen $[0, h]$, of course, but the smaller the initial range, the less computational work has to be done in the first step. But if an initial range different from $[0, h]$ is chosen and if logarithms for this range have been computed (and thus "are known"), means must be found to extend the computation to the remaining range. This was perfectly clear to Napier; he explicitly devised two methods to achieve this, and even compared their respective results. I will not describe these methods here, but will only give the initial ranges needed: $[\frac{1}{2}h, h]$ for the first method, and the considerably smaller interval $[\frac{\sqrt{2}}{2}h, h]$ for the second method. Method 1 is arithmetic, whereas method 2 is trigonometric (remember that Napier deals with logarithms of $SINes$, so a trigonometric approach seems appropriate indeed).

**Accuracy.** Napier has to define some degree of accuracy for his table. Since he wants a table of LN(SIN) with *natural* numbers as logarithms (see above), his criterion refers to the last figure of his logarithms and is given in a kind of statement in § 6: *For in our large numbers, an error which does not exceed unity is insensible and as if it were none.* In modern words, this amounts to saying that the *absolute error in his tables is less than unity* (let me add in passing that this statement will only hold for the initial range). Observe that the true value is not known to Napier! The problem can now be reformulated as follows: starting with $\mathrm{LN}(\mathrm{SIN}(90°)) = 0$ and the properties of LN, the function LN has to be computed in the range [SIN(30°),SIN(90°)] with an absolute error less than 1.

Napier obviously had a clear notion of this task. This becomes especially clear when his treatment of error propagation is considered. Since Napier does not have LN, he cannot do better than undertake a forward analysis of his method of computation. That is, he investigates the propagation of initial errors in order to find the necessary conditions for achieving his aim within the accuracy desired. Since he only has $\mathrm{LN}(h) = 0$ and the properties of LN at his disposal, these properties have now to be investigated. They make use of the four basic operations of arithmetic and of *inequalities.* Napier most ingeniously considers the four basic operations with values that are included between lower and upper bounds (i.e., that are given themselves by *inequalities*). This is what we today call *Interval Arithmetics* – but this is a method usually ascribed to the 20th century!

**Interval Arithmetic.** Although it is used in many places in the *Constructio,* the interval arithmetic investigation of the four basic operations by Napier is rarely mentioned. It is elaborated in the very first paragraphs of the *Constructio:* Let $a$ and $b$ only be known to lie between certain bounds, that is, in an interval:

$$\S\ 7\colon\ 0 < a_{min} < a < a_{max} \quad 0 < b_{min} < b < b_{max}$$

Then one has the following inequalities for the four basic operations of arithmetics:

§ 8: $a_{min} + b_{min} < a + b < a_{max} + b_{max}$

§ 9: $a_{min} \cdot b_{min} < a \cdot b < a_{max} \cdot b_{max}$

§ 10: $a_{min} - -b_{max} < a - b < a_{max} - -b_{min}$

[$a_{min} - b_{max} > 0$ is implicitly required here]

§ 11: $\dfrac{a_{min}}{b_{max}} < \dfrac{a}{b} < \dfrac{a_{max}}{b_{min}}$

Napier in this way analyzes initial errors as well as errors introduced by the operations. He even knows that rounding the bounds thus obtained (if necessary) must be done in such a way that lower bounds are rounded downwards, upper bounds are rounded upwards (§ 12). Thus, Napier anticipates the most advanced and precise form of present-day error analysis.

**Optimal bounds, optimal values.** It is inevitable that Napier at some point in his calculations needs numerical values for variables, but only has the bounds mentioned above. In this case, Napier takes the *arithmetical mean of the bounds,* and considers this mean as the true or at least optimal value. Computing the mean is done as late as possible, in order not to reduce accuracy too early. There are two implicit rules that he seems to obey with this: A number sought to m decimals is optimally bounded, first, if the bounds differ less than one unit of the mth decimal (this of course is not always attainable; let's call these *strong bounds*); or second, if the bounds have been computed with the highest accuracy available within the pre-established number of decimal figures used (*weak bounds*). Taking the arithmetical mean is in most cases considered an optimal choice even today; this could be made more precise, but I refrain from doing this here.

It should be mentioned, however, that the values obtained by Napier in following this procedure are often even more accurate than originally requested by him. To see a glimpse of this, consider the following example: Any number $a$ that is strongly bounded by natural numbers obviously fulfills the condition $n < a < n + 1$ ($n$ a natural number). Taking the mean results in the assignment "true value of $a$" $= n + \frac{1}{2}$. Since in the end only natural numbers are allowed, Napier has the choice of taking either $n$ or $n + 1$ for $a$ – *but none of these values will differ more than 0.5 from the true or optimal value.*

### A reconstruction of Napier's procedure

**The general idea.** Napier's function $y = \mathrm{LN}(x)$ is the *continuous* counterpart of the *discrete* mapping of a geometric sequence to an arithmetic sequence (see above). But into this continuous mapping the discrete mapping can be embedded. This Napier achieves by the following consideration: §36 says that two pairs $(a, b)$ and $(u, v)$, both pairs being in equal proportion, $a : b = u : v$, have the same logarithmic difference:

$$\mathrm{LN}(a) - \mathrm{LN}(b) = \mathrm{LN}(u) - \mathrm{LN}(v)$$

Now in any geometric sequence

$$\{x_k\}_{k=0,1,2,\dots} = \{x_0 q^k\}_{k=0,1,2,\dots}$$

*all* pairs $(x_{k+1}, x_k)_{k=0,1,2,\dots}$ of consecutive numbers are in equal proportion $x_0 q^{k+1} : x_0 q^k = q$, and thus *all* differences $\mathrm{LN}(x_{k+1}) - \mathrm{LN}(x_k)$ are equal. Considering a decreasing geometric sequence $\{x_k\}$ starting at $h$, Napier at once has

$$\begin{aligned} x_0 &= h, \quad x_{k+1} = q \cdot x_k \\ &\qquad (0 < q < 1; k = 0, 1, 2, \dots); \\ x_{k+1} : x_k &= \dots \\ &= x_2 : x_1 = x_1 : x_0 = q < 1; \\ \mathrm{LN}(x_{k+1}) - \mathrm{LN}(x_k) &= \dots = \mathrm{LN}(x_2) - \mathrm{LN}(x_1) \\ &= \mathrm{LN}(x_1) - \mathrm{LN}(x_0) \\ &= \mathrm{LN}(x_1). \end{aligned}$$

The last line obviously implies

$$\mathrm{LN}(x_k) = k \cdot \mathrm{LN}(x_1).$$

It thus suffices to know the value of $\mathrm{LN}(x_1)$ alone in order to obtain *all other* values $\mathrm{LN}(x_k)$ of numbers $x_k$ in the geometric sequence. If in addition one chooses $q$ close to 1, the $x_k$ are close to one another, but one may need many terms of the sequence to reach the other end ($h/2$ for method 1) of the initial range.

On the other hand, logarithms $\mathrm{LN}(x)$ for numbers $x$ not in the sequence $\{x_k\}$ can be obtained (i.e., can be bounded) by § 40, and thus by using interpolation. Since interpolation introduces errors, Napier has to require that his interpolation procedure be accurate within the prescribed degree of accuracy. In other words: it may not be necessary to choose $q$ too close to one (and thus to have to compute an enormous number of terms of the geometric sequence and to add up an equally enormous number of logarithms), but one might be content with a $q$ not so close to 1, *as long as the interpolation scheme of* § 40 *gives results within the accuracy desired.* I will now try to show that this was what Napier really did, before writing the *Constructio* down and before omitting and eliminating any hint – with one possible exception – to this consideration: certainly, one will not find this reasoning in the *Constructio* (remember it employs a purely *deductive style*).

**Interpolation.** Let us assume for the moment that we have chosen a geometric sequence $\{x_k\}$ starting at $h$, having a quotient $q$, $0 < q < 1$, and that $\mathrm{LN}(x_1)$ and thus all $\mathrm{LN}(x_k)$ are known. This is, basically, the situation at § 49 of the *Constructio.* From the interpolation scheme in § 40 we would expect that for any $x$ not in the sequence, Napier would first look for the number $x_k$ in the sequence and nearest to $x$. Then we would expect him to apply § 40 with $a \to x$, $b \to x_k$ or *vice versa,* depending on whether $x < x_k$ or $x_k < x$ (we choose the first case for illustration). This would give him the two bounds for the logarithmic difference $\mathrm{LN}(x) - \mathrm{LN}(x_k)$. We would then expect him to take the arithmetic mean of these two bounds. By adding the value of $\mathrm{LN}(x) - \mathrm{LN}(x_k)$ thus obtained to the already known value of $\mathrm{LN}(x_k)$, Napier would get $\mathrm{LN}(x)$.

But what does Napier's § 50 really say (translated into modern terminology)? "Take $h(x_k - -x)$ and divide this number[7] by the easiest divisor $\bar{x}$, $x \le \bar{x} \le x_k$. Then simply add (or subtract, if $x_k < x$) the result to $\mathrm{LN}(x_k)$." That is, Napier neither computes the two bounds, nor does he take their mean. Instead he does this computation only once, and with a number chosen almost *ad lib,* only satisfying the conditions that it lie between $x$ and $x_k$, and that division is easiest. Why should this procedure lead to accurate results? Here's the only hint that Napier gives: the quotient just described will not *differ by a sensible error from the true difference of the logarithms on account of the nearness of the numbers in the table*[8] *(quorum nullus à vera artificialium differentia errore sensibili differet, propter propinquitatem numerorum Tabulæ*[9]*).* But since for the table mentioned Napier has chosen $q = 0.9995$, the numbers in this table (i.e., in the geometric sequence) are between 2500 and 5000 units apart – which does not seem very near.

**Accuracy of Interpolation.** This procedure should have raised some interest, because in the end it really seems to lead to correct results. But why, then, does

[7]Which is the numerator of each of the two bounds of § 40.

[8]Translation by Macdonald, in: Napier (1889/1966) 36.

[9]Original formulation by Napier, in: Napier (1620) 26–27.

[10]Ayoub (1993) 362 completely misunderstands this crucial procedure when he simply observes that *in fact he* [Napier – JF] *often divides not by a or b* [here $x$ or $x_k$ – JF] *but by some intermediate values to render the arithmetic easier.* Macdonald (in Napier (1889/1966)) does not comment on this procedure, nor does anyone else, as far as I can see.

[11]This is the point that Ayoub seems to make in the beginning of his article [Ayoub (1993)], because it is true that most of what has been written on Napier's logarithms merely describes in more or less modern terms what Napier did, but not why he did it that way. Alas, Ayoub also repeats this kind of descriptive translation time and again, based mainly on Macdonald's sparse and at the same time not very reliable comments and interpretations in Napier (1889/1966).

the division with any $\bar{x},\ x \le \bar{x} \le x_k$ work[10]? I propose the following reconstruction, which will not only describe Napier's choice for $q$, but will explain it[11].

The two bounds of § 40 delimit the difference LN$(x)-$ LN$(x_k)$. Assuming that LN$(x_k)$ is known within the accuracy required by Napier, i.e., with an absolute error less than 1, then the difference to be added in order to obtain LN$(x)$ must not have an error greater than 1, too. (In fact, both numbers should be known even more precisely; namely, with an absolute error less than 0.5, because the respective errors during addition also might add up, if they were both in the same direction. Napier seems to have known this, and indeed his numbers are more precise, as hinted above.) In other words, the difference *has to be strongly bounded* by these two bounds, which in turn means that the difference between the bounds must not exceed unity. That is,

$$\frac{h(x_k - x)^2}{x \cdot x_k} < 1$$

.

**The condition for** $q$. Let us now consider this expression in more detail (but I will discuss *in extenso* only the case $x < x_k$, as before). Since the sequence $\{x_k\}$ is decreasing, we have $x_{k+1} < x < x_k$. Then the distance $x_k - x$ obeys the inequality

$$0 < x_k - x \le \frac{1}{2}(1-q)x_k$$

.

To see this, observe that $x_k - x_{k+1} = (1 - -q)x_k$ is the distance from $x_k$ to $x_{k+1}$, and as $x$ is nearest to $x_k$, the distance from $x$ to $x_k$ is at most half of the distance $x_k - x_{k+1}$ (because otherwise $x_{k+1}$ would be nearest to $x$). A reformulation results in the second inequality

$$\frac{1}{2}(1+q)x_k \le x < x_k$$

.

The accuracy requirement above is fulfilled *a fortiori,* if the numerator of the fraction is increased by the first inequality, and if the denominator is decreased by the second inequality, before the fraction itself is bounded by 1:

$$\frac{h(x_k - x)^2}{x \cdot x_k} \le \frac{h\frac{1}{4}(1-q)^2 x_k^2}{\frac{1}{2}(1+q)x_k \cdot x_k} = \frac{h(1-q)^2}{2(1+q)} < 1$$

.

This quadratic inequality holds for $0.99936764... < q < 1$ (the case $x_k < x < x_{k--1}$ leads to a linear inequality with solution $0.99936794 \cdots < q < 1$). The first easy–to–handle $q$ to fulfill these conditions is exactly the value that Napier chose: $q = 0.9995$. This is, I think, the explanation for his choice of $q$, and it has been *deduced* from the requirement that interpolated numbers must have the same accuracy as directly computed numbers.

**The number of terms needed.** The condition for q was derived under the assumption that all LN$(x_k)$ are known with the accuracy required. Since LN$(x_k) = k\cdot$LN$(x_1)$, this in turn raises the question of the accuracy required for LN$(x_1)$. The expression $x_k = q^k x_0 = q^k h$ suggests considering $q^k$ with $q = 0.9995$:

$$q^0 = 1.0000, q^1 = 0.9995, q^2 \approx 0.9990, \ \ldots\ q^{20} \approx 0.9900$$

With $p = 0.99 \approx q^{20}$, Napier advances more rapidly[12]:

$$p^1 = 0.99, \quad p^2 \approx 0.98, \ \ldots, \quad p^{69} \approx 0.5,$$

more precisely: $p^{69} < 0.5 < p^{68}$. Therefore $h/2 \approx p^{69}h \approx (q^{20})^{69}h = q^{1380}h$; a geometric sequence with $q = 0.9995$ has reached $h/2$ (the other end of the initial range of method 1, see above) with its 1380th term and thus fills the interval of computation $[\frac{1}{2}, h]$ with points of reference that are dense enough for accurate interpolation.

**The accuracy needed for** LN$(x_1)$. The accuracy of the last term's logarithm, LN$(x_{1380}) \approx$ LN$(h/2)$, is worst among all LN$(x_k)$. Since LN$(x_{1380})$ = 1380LN$(x_1)$, and since LN-values are natural numbers, LN$(x_1)$ = LN(9995000) has to be computed with accuracy at least $1/1380 \approx 0.0007$ for the final absolute error to be less than 1. But § 40 only gives us $5000 <$ LN(9995000) $< 5002.50125$, which is far from the accuracy required. There are two possible solutions to this problem: either Napier could *iterate* the procedure, using [9995000, 10000000] as a new interval of computation and 0.0007 as the accuracy required therein, thus looking for a new condition for $q$ in this interval – or he could have a look at the problem from the other side. I think he followed this way.

LN(9999999). Computation of the table of LN$(SIN)$ requires evaluation of LN at 5400 *natural* numbers $x = SIN(a)$; it seems probable to me that Napier therefore concentrated on natural numbers. He knows that $h = 10^7$ has LN$(h) = 0$; the natural number nearest to $h$ in $[0, h]$ is $h - 1 = 9999999$. This number can be viewed as the term $v_1$ of another geometric sequence, starting at $v_0 = h$ with quotient $r = 0.9999999$, $v_{k+1} = 0.9999999v_k,\ k = 0, 1, 2, \ldots.$ Then § 29 gives us now the inequality

$$\begin{aligned} 1 &= h - 9999999 < \mathrm{LN}(9999999) < h\frac{(h-9999999)}{9999999} \\ &= 1.0000001\ldots, \end{aligned}$$

and these are the sharpest bounds attainable from § 29 for any LN$(x)$, if $x$ is limited to natural numbers. The difference between the bounds of this special LN–value is one unit of the 7th decimal; and *this is the reason why Napier initially computes all his LN–values to 7 decimals.*

[12] To see that Napier really made use of this acceleration, it suffices to have a look at the three tables that he subsequently forms. They clearly show the points where he switched to a new quotient – this, of course, with the intention of making the arithmetic easier.

**Joining the sequences.** Napier now considers $r^k$ (because $v_k = r^k v_0 = r^k h$):

$$r^0 = 1.0000000, \quad r^1 = 0.9999999,$$
$$r^2 \approx 0.9999998, \ldots r^{100} \approx 0.9999900$$

With $s = 0.99999 \approx r^{100}$, he may advance more rapidly:

$$s^1 = 0.99999, \quad s^2 \approx 0.99998, \ldots \quad s^{50} \approx 0.99950,$$

more precisely: $s^{51} < 0.9995 < s^{50}$. Therefore the sequence $\{v_k\}$ has joined the sequence $\{x_k\}$ with its 5000th term:

$$s^{50}h \approx (r^{100})^{50}h = r^{5000}h = v_{5000} \approx 9995000 = x_1.$$

Since $\mathrm{LN}(v_k) = k \cdot \mathrm{LN}(v_1)$, the accuracy of $\mathrm{LN}(v_{5000})$ decreases to $5000 \cdot 0.0000001 = 0.0005$, but since $0.0005 < 0.0007$, the accuracy needed for LN(9995000) is now guaranteed.

## Epilogue

There is still a lot that could be said about the further execution of Napier's program, because the real numerical work has not even begun at this point of the story. But I hope some of the aspects that Napier had to consider before setting to work have come to light. To my knowledge, the task that Napier undertook is unprecedented in the history of mathematics. The numerical and computational work that he had before him, once his approach had been found, still was tremendous and probably took him some decades to execute. Undoubtedly he made good use of his own Napier's bones, but this is another aspect of the matter. The theoretical work that had to be done before the numerical work could start was immense, too.

Although Napier presented his kinematic concept of geometrically and arithmetically moving points on a straight line – which eventually leads to his function LN – with some clarity, he carefully avoided mentioning anything that might be considered superfluous when it came to the construction of his tables. That is why one has to read authors like Napier with extreme care, because much important information can only be found between the lines. If this has become sufficiently clear to the reader, my aim is achieved.

## Bibliography

Ayoub, Raymond: "What Is a Napierian Logarithm?", *American Mathematical Monthly* 100 (1993) 351-364 [April 1993].

Ayoub, Raymond: "Napier and the Invention of Logarithms", *Journal of the Oughtred Society,* Vol. 3, No. 2 [September 1994] 7-13.

Fischer, Joachim: "Looking 'behind' the Slide Rule: How did Napier compute his Logarithms?", *Proceedings of [the] Third International Meeting of Slide Rule Collectors,* September 12, 1997, Faber-Castell Castle, Stein/Nürnberg, 8-18.

Glaisher, James Whitbread Lee: Articles "Logarithm", "Napier", *The Encyclopædia Britannica,* New York: The Encyclopædia Britannica Company 1911.

Knott, C. G. (ed.): *Napier Tercentenary Memorial Volume,* London, Royal Society of Edinburgh 1915.

Macdonald, William Rae: Article "Napier", *The Oxford English Dictionary,* Oxford: Oxford University Press 1933.

Napier, John, *Mirifici logarithmorum canonis constructio [...],* Edinburgh, A. Hart 1619.

Napier, John, *Mirifici logarithmorum canonis constructio (qui et tabula artificialis ab authore deinceps appellatur) eorumque ad naturales ipsorum numeros habitudines,* Lyon, B. Vincent 1620.

Napier, John, *The Construction of the Wonderful Canon of Logarithms* [English translation by William Rae Macdonald]. Reprinted 1966 for Dawsons of Pall Mall, London.

Naux, Charles, *Histoire des logarithmes de Neper à Euler,* 2 vols.; Paris: Blanchard 1966 [vol. I] /1971 [vol. II]; here vol. I, pp. 34-91.

Otnes, Robert K., "An Example of Napierian Logarithms", *Journal of the Oughtred Society,* Vol. 4, No. 1 (March 1995) 46-47.

Struik, Dirk J., *A Source Book in Mathematics, 1200-1800,* Princeton: Princeton University Press 1969 (paperback edition 1986); section I-4, pp. 11-21.