JOS Plus

The Boonshaft and Fuchs Direct Reading Frequency Response Slide Rule Revisited

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"If at first you don't succeed, the hell with it – or not" Anonymous

*Several years ago, the Journal of the Oughtred Society published my article on the Boonshaft and Fuchs slide rule (Volume 20, No. 2 Fall 2011). In retrospect, my article was perhaps overly complex, and did not stress the true usefulness of this slide rule. Bob Otnes lent me his Boonshaft and Fuchs slide rule for study, but unfortunately the instructions were missing. I was an analog circuit designer at various naval research centers at China Lake for ~38 years and had tried to make this rule fit my needs in electronic filter design. Fortunately, I recently obtained a Boonshaft and Fuchs with some instructions which indicated this slide rule was a 'Direct Reading Frequency Response Slide Rule' for the design of Servo feedback control systems. I was almost right because the filter design equations the slide rule helps to solve are similar. Here is a brief discussion on what problems this slide rule was designed to solve, illustrated with several simple examples. The Appendix reprint that came with my slide rule is an excellent overview of the slide rule's purpose. The instructions are included in **JOS Plus**[1] (I have also included the description in the Appendix, which you can print and study at your leisure).*

The Boonshaft and Fuchs Direct Reading Frequency Response slide rule (See Figure 1) was invented by Jens R. Jensen of the Danish Technical University in Copenhagen. Jensen teamed up with M. Drost Larsen to form the Servo Calculation Company, who manufactured these slide rules. They were then sold to Boonshaft and Fuchs Inc., who were the sole distributors for the US and Canada. Boonshaft and Fuchs started selling the slide rule around June 1959.

At 15.75 inches (40 cm) in length and 5 inches (12.7 cm) in width, the slide rule really is quite large, and has spaces for eleven slides. The basic version which Bob Otnes lent me came with twenty-five slides, and an easily reversible cursor to read the reverse scale readings. The slides are used to calculate the transfer function gain in dB and the phase angle for Servo Feedback Control systems. The basic slide rule came in a leather case (see pictures in the JOS Plus) and was priced, in 1959, at $170, an expense well out of reach for most engineers. The primary slide is the frequency (u) scale which has six decades, in twenty equally spaced divisions from 0.001 to 1000 (See Figure 2). The "basic" version has twenty-four individual transfer function slides with the gain on one side, and the phase angle on the other. The second version has the same slide rule frame and cursor, but came with seventy individual slides. The slides include transfer functions, gain in dB (dB = 20Log G), and phase angle. The slide rule designers used centibels, so the gain readings must be divided by 10 to obtain the gain in dB.

The frequency (u) scale is an expanded six-decade logarithmic D scale with twenty frequencies equally spaced at 0.05 increments on the Log u (an expanded six-decade L scale) scale where $u = 10^{(\text{Log u reading})}$. The **black** readings are positive and the **red** are negative.

Several years ago Bob Otnes lent me his "basic" version (twenty-five slides) for study, and the results were published in the Journal of the Oughtred Society (20:2, Fall 2011). I was an electronic engineer (analog circuit design) and tried to make the Boonshaft and Fuchs fit my previous engineering needs in electronic filter design. I was almost right because the slide rule can actually be used for electronic filter design despite the fact that the slide rule was originally designed to aid in the solution of Servo feedback control transfer functions. I recently obtained a version II slide rule with seventy slides, at a cost of certainly more than the $170 for the basic version. I purchased this version because instructions were included, enabling me to determine how close my previous analysis had been. What follows is a basic discussion on what the slide rule does, with several simple examples.

I really did not think I would like such a large slide rule, but during the past four months, I have found that as an aid in the solution of complex Servo Feedback Control transfer functions, the intended purpose of the slide rule, the Boonshaft and Fuchs does an excellent job. We begin by answering this question: just what is a Servo feedback controller?

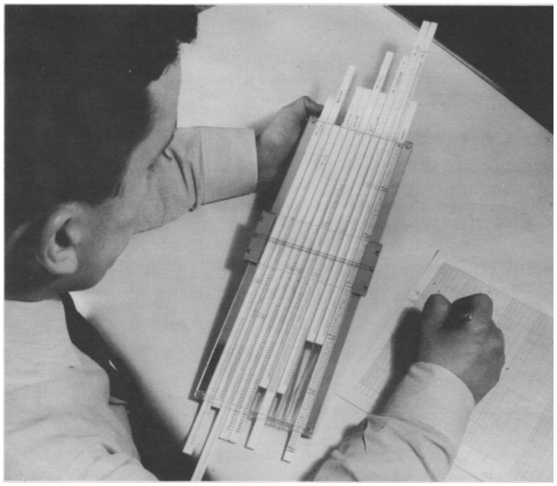
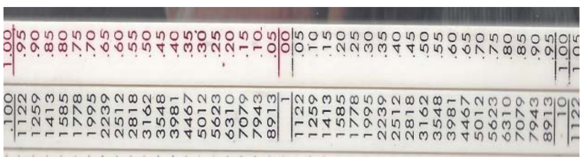


FIGURE 1. The Boonshaft and Fuchs Slide Rule
The cursor is easily removable to read the numbers on the reverse side.



	Log u	u
	-1	0.100
	-0.6	0.2512
Log u	0.0	1
	0.3	1.995
Frequency u	0.6	3.981
→	1.00	10

FIGURE 2. Frequency u and Log u Scales and Slide Rule Readings

Basic Servo Feedback Control

The operation of the basic Servo feedback system shown in Figure 3 is quite straightforward. The user, turns a potentiometer θ_i degrees giving V_i volts at the input of the differencing amplifier. This voltage is amplified and drives the Servo motor, whose output angle, θ_o, is a function of the output P of the power amplifier. The Servo motor has a circuit that gives a voltage, V_o, that is a function of the Servo motor output angle θ_o. The voltage V_o representing θ_o is subtracted from V_i, the voltage representing the input angle θ_i to give the error voltage V_e ($V_e = V_i - V_o$). This scheme is the foundation of negative feedback: as $V_e \rightarrow 0$, $V_o = V_i$, and the Servo motor output angle is θ_o. The output phase angle, θ_o, divided by the input voltage, V_i, is the system gain, and we are interested in the gain and phase angle variation as a function of the input sinusoidal frequency variation V_i.

I begin with a brief discussion of Bode (pronounced Boh-dee) plots, which display both gain in dB and phase angle versus frequency on a Log scale. We are simply solving for the hypotenuse and phase angle of a right triangle, while noting that in vector notation, the y-axis is labeled +j (90° up) and 1/j = -j (-90° down). Consider the basic right triangle shown in Figure 4. Letting x = 1, point P may be given as 1+jy (y at 90° with respect to the x axis), or P = r∢θ. The hypotenuse $r = \sqrt{1+y^2}$ and the phase angle $\theta = \tan^{-1} y$. j and 1/j = -j simply represent a- +/-90° rotation of the y axis.

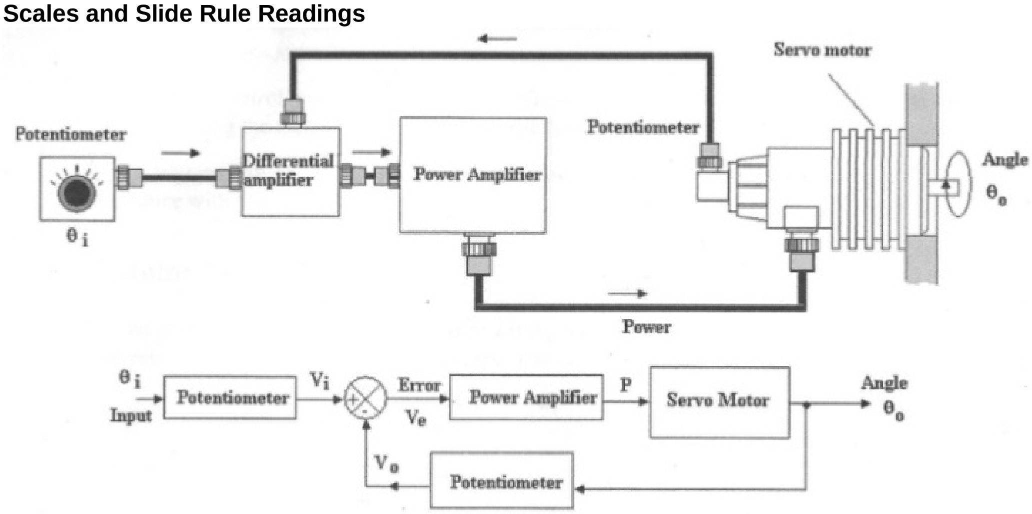


FIGURE 3. Basic Servo Feedback Control

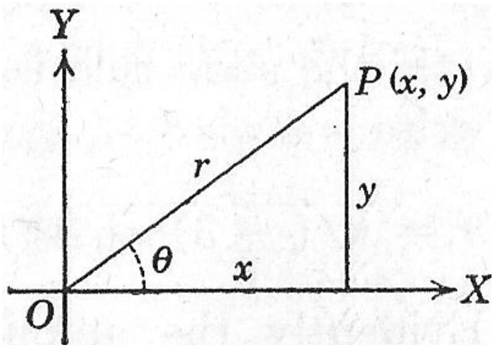


FIGURE 4. Right Triangle
x = 1, P = r∡ θ; if θ is negative

Engineers like to express gain (r) in dB (dB = 20Log(r). Thus

$$r(\text{gain})_{dB} = 20\text{Log}(\sqrt{1 + y^2}) = 10\text{Log}(1 + y^2) = \text{Gain}_{dB}$$
$$\theta = \tan^{-1} y$$

The terms Gain and phase angle refer to the gain, G, and phase angle, θ of Servo control system components, $G_{db}\measuredangle\theta$.

In 1938 Hendrik Wade Bode published his concept of plotting both the gain in dB, and phase angle versus frequency on a Log scale. His Bode plots are still the norm for transfer function analysis today.

The Transfer Function $1 + j(\omega/\omega_0)$

Consider the transfer function $1 + j(\omega/\omega_0)$. Although the slide rule uses u for the frequency, the same right triangle (See Figure 4) can be used with ω/ω_0 along the y(j) axis, where ω_0 is just a constant related to the transfer function as discussed below. Engineers like to name things, and we usually define transfer functions as H(jω). The solution for our transfer function can be written as

$$H(j\omega) = 1 + j(\omega/\omega_0)$$
$$\text{Gain}_{dB} = 20\ \text{Log}|H(j\omega)| = 10\ \text{Log}\ [1 + (\omega/\omega_0)^2]$$
$$\theta = \tan^{-1}(\omega/\omega_0) = \text{Gain}_{db}\measuredangle\theta$$

For low frequencies, where $\omega \ll \omega_0$, the gain is $10\text{Log}\ (1^2) = 0_{dB}$ with a phase angle $\theta = \tan^{-1}(0) = 0^\circ$. At high frequencies, where $\omega > \omega_0$, the gain increases to $\text{Gain}_{dB} = 10\ \text{Log}[(\omega/\omega_0)^2]$, and the phase angle $\theta = \tan^{-1}(\omega/\omega_0) = +90^\circ$. What about when $\omega = \omega_0$? In that case, $\text{Gain}_{dB} = 10\ \text{Log}\ [1 + (1)^2] = 10\text{Log}(2) = +3.01\text{dB}$ with a phase angle $\theta = \tan^{-1}(1\ ) = +45^o$, so that $H(j\omega) = +3_{dB}\measuredangle{+45^o}$. The $+3_{dB}$ break frequency ω_0 is an important concept as we will see. Bode plotted the Gain_{db} and the phase angle versus frequency on a log scale. As shown in Figure 5 the Boonshaft and Fuchs has numerous individual transfer function slides to help in the plotting of the Bode plot.

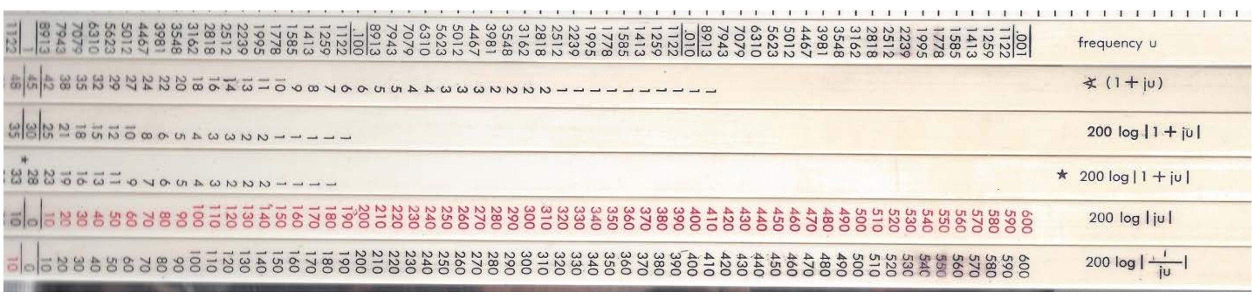


FIGURE 5. First Order (jω) Transfer Function ω/ω_0 (slide rule u/u_0) Slides

Twenty-three transfer function slides are provided. The transfer functions slides have the gain, 200Log (remember to divide the readings by 10 for dB since the slide rule uses centibels) on one side and the phase angle, ∡ on the other. The slides marked *200Log are designed to provide Gain_{db} and phase angle readings in 0.05 steps starting at 0.025 (See Figure 2), and helps fill in the gaps on the 200Log scales if necessary. The black readings are positive and the red are negative.

The Gain_{dB} and phase angle for the transfer function $1/(1 + j(\omega/\omega_0))] = -\text{Gain}_{dB}\measuredangle{-\theta}$ are easily solved as we discussed above ($\text{Gain}_{dB} = -10\text{Log}[(1+ j(\omega/\omega_0)]$ and $\theta = -\tan^{-1}(\omega/\omega_0)$. The only difference is that now the Gain_{dB} and phase angle are negative (Gain_{dB} of -3dB with a phase angle = -45°). Table 1 shows the ideal Bode plot for $\omega_0 = 1$. You will soon see how easy any ω/ω_0 value can be handled, as long as you remember that the Boonshaft and Fuchs uses u for the frequency ω.

We now discuss a simple example, the First Order, jω, transfer function.

$$\mathbf{H(j\omega) = 2(1 + j\omega/0.4)/(1 + j\omega/8)}$$

This transfer function has two individual transfer functions $1 + j(\omega/0.4)$ with a +3dB break frequency $\omega_0 = 0.4$, and $1/[1 + j(\omega/8)]$, with a -3dB break frequency $\omega_0 = 8$. To use the slide rule, place +3dB (remember we must divide by 10 for dB) on the 200Log[1+j(u)] slide over $\omega = 0.4$ on the frequency slide, and -3dB on the 200Log1/[1+j(u)] slide. (JOS Plus has an excellent brief discussion on setting up the slide rule). Now all we have to do is read the Gain_{dB} and phase angle for each frequency to build a table of Gain_{dB} and phase angle versus frequency. The results are given in Table 2. Time is needed to generate the table, but the slide rule makes generating the table much easier. The slide rule results are in excellent agreement with the computer-generated Bode plot.

One of the primary functions of the Boonshaft and Fuchs is to help to determine if a design is stable.

Stability

Let us consider the Servo Feedback controller shown in Figure 6. We are all familiar with such a controller, namely the speed controller on our car. Simply set the speed control, v_{ref}, to the desired speed. A voltage proportional to the wheel speed, v_s, is subtracted from v_{ref} in the differencing amplifier, $v_{error} = v_{ref} – v_s$ and the error voltage ultimately drives the wheel speed Servo motor. The wheel speed is determined by the wheel speed sensor resulting in the feedback voltage v_s. Ideally, the error voltage, v_{error}, is driven to 0 so that our speed remains stable and constant: $v_{error} = v_{ref} - v_s \rightarrow 0$ and so $v_{errer} = v_{ref}$. Anyone who has designed feedback systems, be they Servo or electronic systems, knows what can, and all too often does happen. The controller may become unstable, resulting in oscillations, or in even worse behavior. In our example, the speed would keep increasing. What happens if v_s is negative (a -180° phase shift)? The error voltage is positive $v_{ref} - (-v_s) = v_{error} = v_{ref} + v_s$. Now v_{error} increases, instead of decreasing, which is certainly an undesirable outcome. Consider the gain from v_{error} to v_s, which is called the Loop Gain. If the Loop Gain transfer function has a Gain_{db} greater than 0_{db} at a phase angle of -180° or greater, we have an unstable system. The Boonshaft and Fuchs now comes to our aid. Although this instability is usually fairly easy to correct, knowing that we have a problem before committing to hardware is always preferable. Determining system stability is one of the primary advantages of the slide rule. Knowing the Loop Gain transfer function, set up the slide rule and look for a phase angle -180° or greater, turn the slide rule over and calculate the Gain in dB. If this Gain is greater than 0_{dB} we know we have a problem.

Table 3 shows the results for an unstable Servo Controller. Note that for a frequency $\omega = 2.2$ (slide rule reading u = 2.239) the Loop Gain is +10.4 dB for a phase angle $\theta = 180°$, forcing us back to the drawing board. I will close with a brief discussion on Second Order transfer functions, $(j\omega/\omega_0)^2$

$$\mathbf{H(j\omega) = 1/[1+2\zeta j(\omega/\omega_0)+(j\omega/\omega_0)^2]}$$

The second Boonshaft version has two leather pouches, each containing twenty-six slides to aid in the solution of Second Order transfer functions. Some of these slides are shown in Figure 7.

The mathematics can be quite complicated, so I will forgo any discussion. Table 4 shows the results for $\omega_0 = 2$ and $\zeta = 0.3$. As expected, the slide rule results are in excellent agreement with the computer-generated plots.

In conclusion, I note that the Boonshaft and Fuchs Direct Reading Frequency Response Slide Rule is easy to use, once the transfer function is known. Although my examples are simple, the Boonshaft and Fuchs easily handles more complex transfer functions (See Figure 7). The Appendix in **JOS Plus** is an excellent overview that came with the slide rule. A more in-depth discussion is given in **JOS Plus**.

My friend Dr. Sam Galeb, PhD Nuclear Engineering, provided the transfer function graphs.

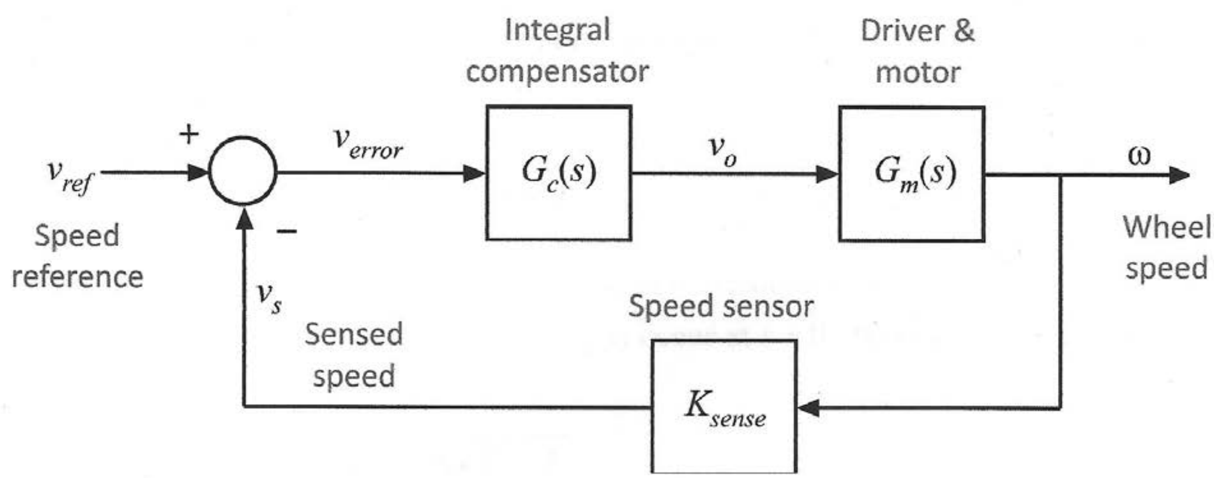


FIGURE 6. Basic Speed Control

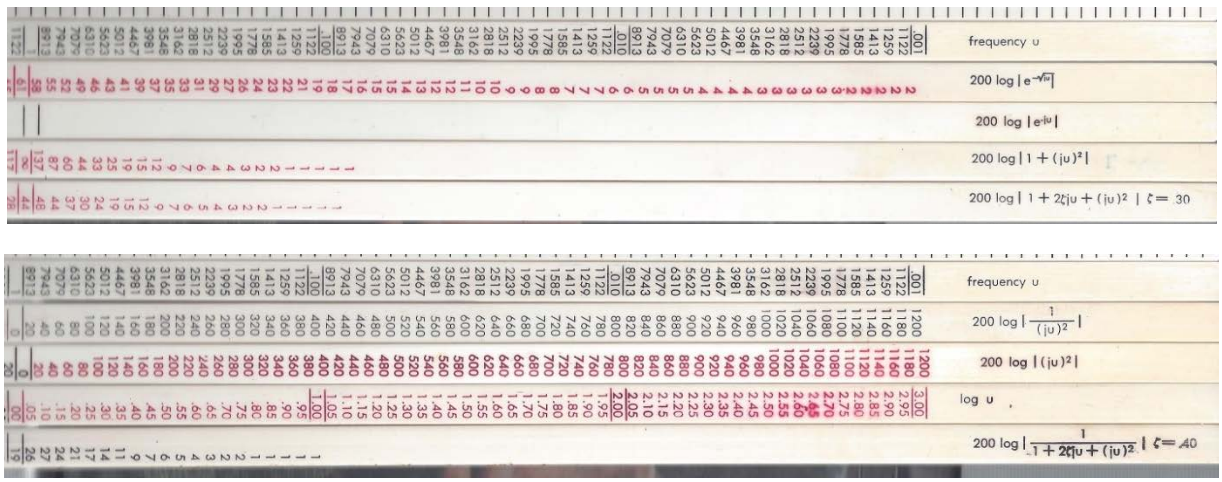


FIGURE 7. Second Order Transfer Function Slides

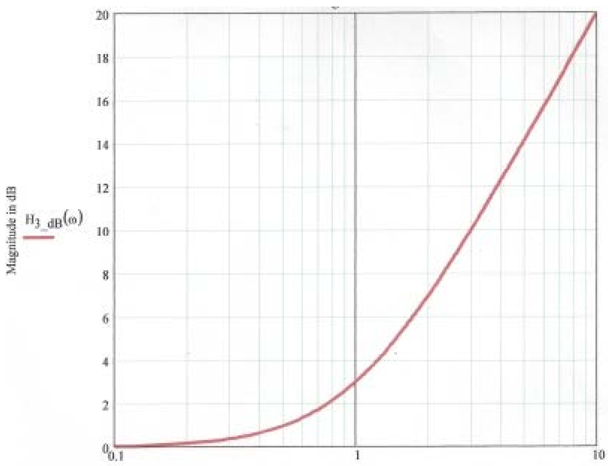
Twenty slides are provided for $H(j\omega) = 1/[1+2\zeta j\omega/\omega_0+(j\omega/\omega_0)^2]$ and $H(j\omega) = [1+2\zeta j\omega/\omega_0+(j\omega/\omega_0)^2]$. They are a function of the "damping factor", ζ, and range from 0.025, 0.05 to 0.95 in 0.05 steps.

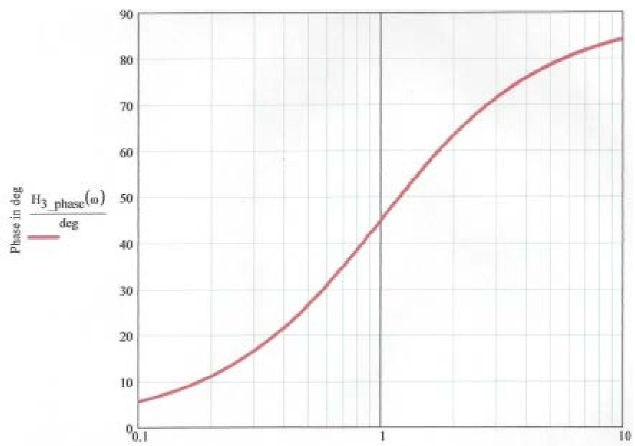
TABLE 1. The Boonshaft and Fuchs and the Normalized Bode Plot, $1 + j(\omega/\omega_0)$

$1 + j(\omega/\omega_0)$ represents a right triangle, with the x axis = 1 and the y axis = ω/ω_0 at 90° (the j, y axis is j that is equal to +90°). Thus, the hypotenuse or magnitude, also called the gain, $|H(j\omega)|$ is a function of ω (frequency) with a phase angle θ°. The result is a Magnitude (Gain), with a phase angle θ.

Magnitude = $\sqrt{1 + (\omega/\omega_0)^2}$; Magnitude, in dB = $20\text{Log}\sqrt{1 + (\omega/\omega_0)^2} = 10\text{Log}[1+ (\omega/\omega_0)^2]$ $\theta = \tan^{-1}(\omega/\omega_0)$

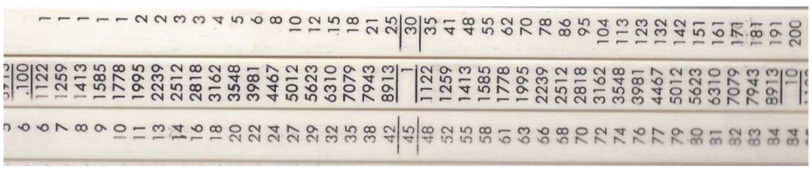
Magnitude Plot **Phase Plot**





ω/ω_0 ω/ω_0

Black numbers for positive values; ω = u



200Log(1 + jω)

Frequency u →

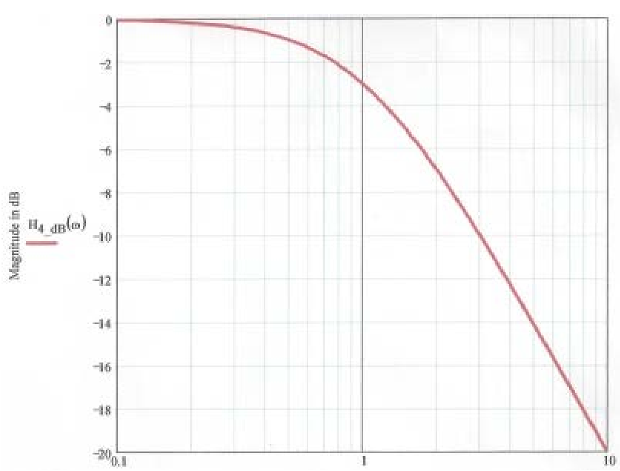
∢200Log(1 + jω)

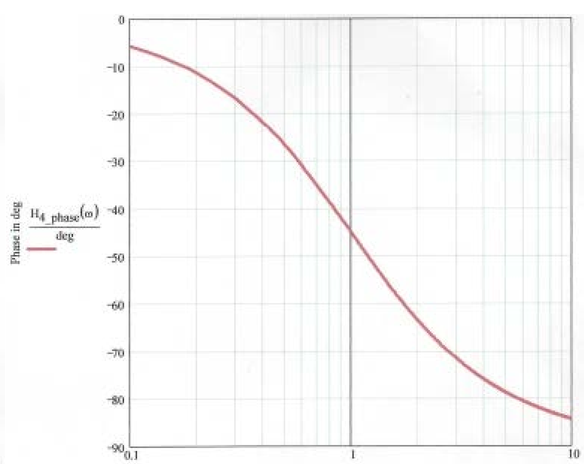
1 /(1 + j (ω/ω₀))

The denominator, 1 + j(ω/ω_0), represents a right triangle, with the x axis = 1 and the y axis = ω/ω_0 at -90°. The triangle hypotenuse or magnitude |H(jω)| determines the gain -20 Log|H(jω)| as a function of frequency ω. From an analysis of this triangle, both the Magnitude or Gain, and the phase angle θ are obtained as

Magnitude = $1/\sqrt{1+(\omega/\omega_0)^2}$; Magnitude, in dB = $-20\text{Log}\sqrt{1+(\omega/\omega_0)^2}$= $-10\text{Log}[1+(\omega/\omega_0)^2]$ $\theta = -\tan^{-1}(\omega/\omega_0)$

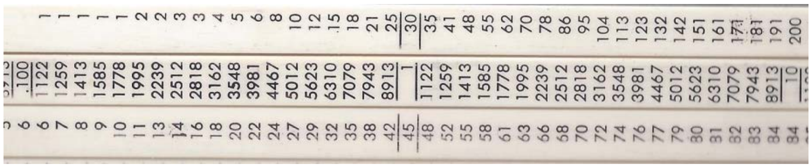
Magnitude Plot **Phase Plot**





ω/ω_0 ω/ω_0

Red numbers for negative values; ω = u



200Log(1 + jω)

Frequency u →

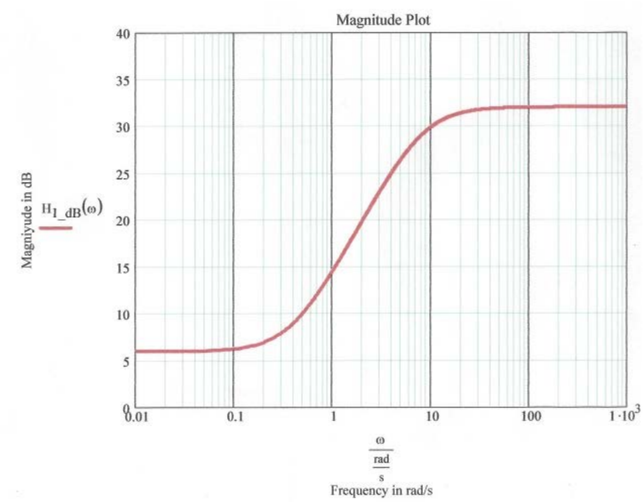
∡200Log(1 + jω)

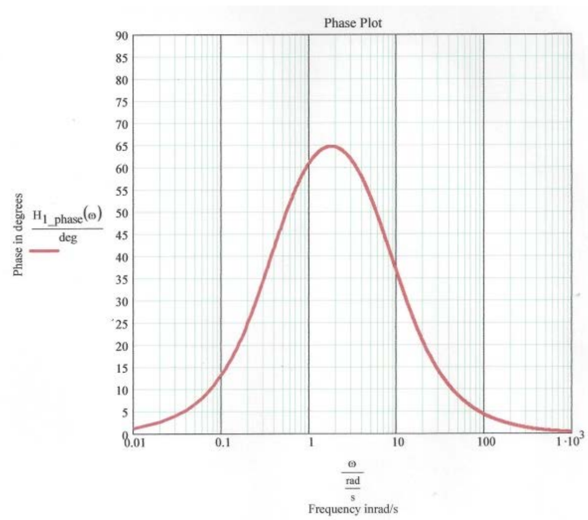
TABLE 2. First Order Transfer Function, j(ω/ω$_0$)

H(jω) = 2(1 + jω/0.4)/(1 + jω/8)

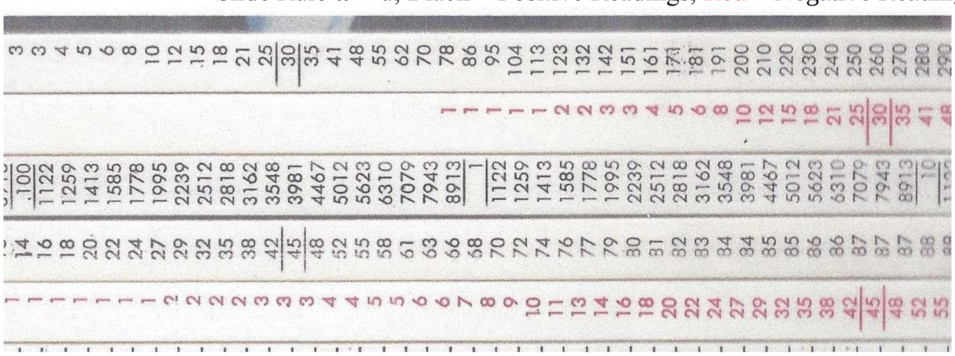
Magnitude, dB = 20Log2(6dB) + 10Log[1 + (ω/0.4)^2] – 10Log[1 + (ω/8)^2)

θ = tan^{-1}(ω/0.4) – tan^{-1} (ω/8)





Slide Rule ω = u; Black = Positive Readings, Red = Negative Readings



200Log[1+j(u/0.4)]

200log[1/1+j(u/8)]

Frequency, u→

∡ [1+j(u/0.4)]

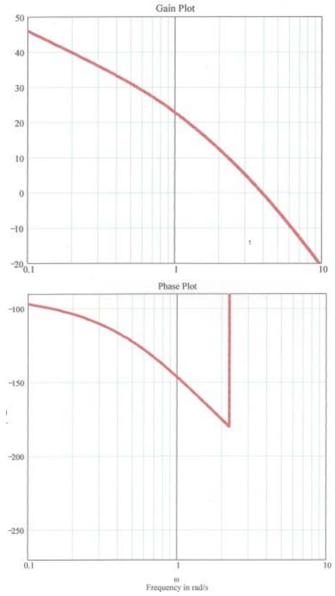
∡ [1/1+j(u/8)]

Slide Rule Readings (Remember to Divide 200Log by 10 for dB)

Frequency	Magnitude, dB				Phase Angle θ		
ω (u slide rule)	20Log2	10Log[1 + j(ω/0.4)^2]	10Log[1 + j(ω/8)^2]	Total	tan^{-1}(ω/0.4)	– tan^{-1} (ω/8)	Total
0.1 (0.1)	+6	+0.3	0	+6.3	+14	-1	+13
0.4 (0.3981)	+6	+3	0	+9	+45	-3	+42
1 (1)	+6	+8.6	-0.1	+14.5	+68	-7	+61
2 (1.995)	+6	+14.2	-0.3	+19.9	+79	-14	+65
4 (3.981)	+6	+20	-1	+25	+84	-28	+56
10 (10)	+6	+28	-4.1	+29.9	+88	-52	+36

TABLE 3. Stability

Loop Gain G (jω) = 20 /[(jω/1)(1 + j ω/1)(1 + j ω/5)]



Loop Gain = 20 /[(jω/1)(1 + j ω/1)(1 + j ω/5)]

G (jω)db = 20Log20(+26dB) – 20Log(ω/1) – 10Log [1 + (ω/1)2]

-10Log[1 + (ω/5)2]

LLoo

θ = -90°(1/j = -j) – tan-1(ω/1) – tan-1(ω/5)

The Loop Gain is +10.4dB at a phase shift = -180°, and the system is unstable

Loop Gain = 20 /[(jω/1)(1 + j ω/1)(1 + j ω/5)]

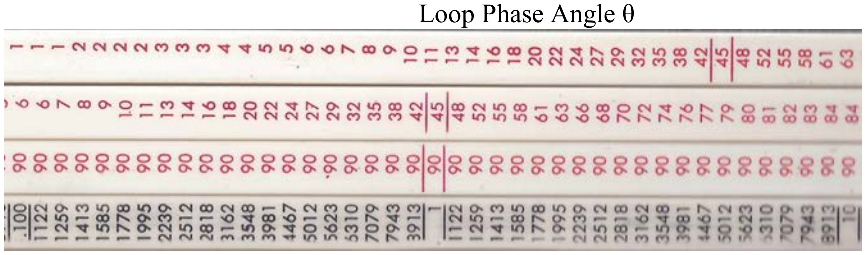
$G(j\omega)_{db}$ = 20Log20(+26dB) – 20Log(ω/1) – 10Log [1 + $(\omega/1)^2$] -10Log[1 + $(\omega/5)^2$]

$\theta = -90^{\circ}(1/j = -j) - \tan^{-1}(\omega/1) - \tan^{-1}(\omega/5)$

The Loop Gain is +10.4dB at a phase shift = -180°, and the system is unstable

The computer program does not like angles greater than -180°; however, this does illustrate the effect of a Loop Gain greater than 0dB for a phase angle of -180°.

Loop Phase Angle θ



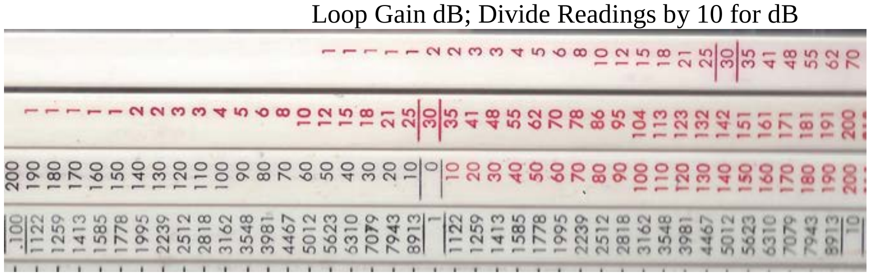
∡ [1/1+j(u/5)]

∡ [1/1+j(u)]

∡ [1/j(u)]

Frequency, u →

Loop Gain dB; Divide Readings by 10 for dB



200log[1/1+j(u/5)]

200log[1/1+j(u)]

200log[1/j(u)]

Frequency, u →

Slide Rule Readings; Red are Negative Readings

Frequency	Loop Gain dB 20Log20 = 26dB				Loop Phase Angle θ			
ω (u slide Rule)	1/j(ω/1)	1/(1+jω/1)	1/(1+jω/5)	Total	1/jω	1/(1+jω/1)	1/(1+jω/5)	Total
0.2 (0.1995)	+14	-0.2	0	+39.8	-90	-11	-2	-103
0.7 (0.7079)	+3	-1.8	-0.1	+24.1	-90	-35	-8	-133
1 (1)	0	-3	-0.2	+22.8	-90	-45	-11	-146
2.2(2.239)	-7	-7.8	-0.8	+10.3	-90	-66	-24	-180
4 (3.981)	-12	-12.3	-2.1	-0.038	-90	-76	-38	-204
5 (5.012)	-14	-14.2	-3	-5.2	-90	-79	-45	-214
10 (10)	-20	-20	-7	-21	-90	-84	-63	-237

TABLE 4. Second Order Transfer Functions, $j(\omega/\omega_0)^2$

General

$H(j\omega) = 1/[1+2\zeta j\omega/\omega_0+(j\omega/\omega_0)^2]$

Knowing ω_0, $x \equiv 2\zeta/\omega_0$, $\zeta = x\ \omega_0/2$ so that $H(\text{at } \omega_0) = H(j\omega_0) = 1/2j\zeta = 1/2\zeta$ ($\theta = -90^\circ$ using $1/j = -90^\circ$))

The maximum value for $H(j\omega)$ is a function of ζ, so that $H_{Max}(j\omega) = 1/[2\zeta\sqrt{(1 - \zeta^2)}]$

and the frequency for $H_{Max}(j\omega)$, $\omega(H_{Max}(j\omega)) = \omega_0\sqrt{(1 - 2\zeta^2)}$

Example; $\omega_0 = 2$

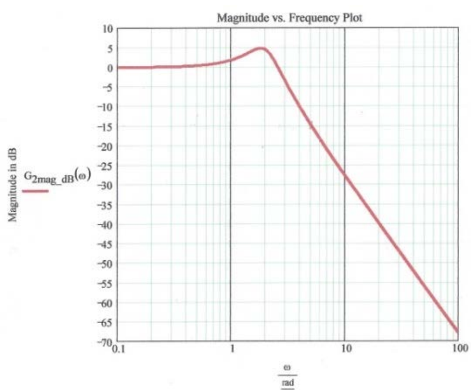
$H(j\omega) = 1/[1+0.3j\omega/\omega_0+(j\omega/2)^2]$

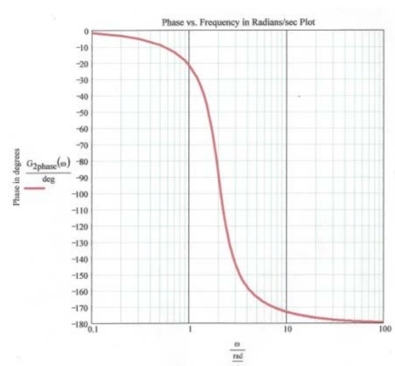
$\omega_0 = 2$; $x= 0.3$; $\zeta = x\ \omega_0/2 = 0.3(2)/2 = 0.3$

$H(j\omega_0) = 1/2j\zeta =1/2(0.3)j = -j1.66$ (Gain = 20 Log $|H(j\omega_0)|$= 4.4dB) ($\theta = -90^\circ$ using $1/j = -90^\circ$)

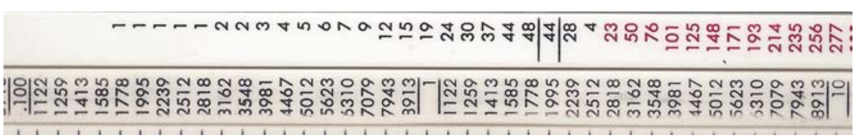
$H_{Max}(j\omega) = 1/[2\zeta\sqrt{(1 - \zeta^2)}] = 1/[2(0.3)\ \sqrt{[0.6(1 - (0.3^2)]} = 1.75$ (4.8dB)

$\omega(H_{Max}(j\omega)) = \omega_0\sqrt{(1 - 2\zeta^2)} = 2\sqrt{[1 - 2(0.3)^2]} = 1.81$





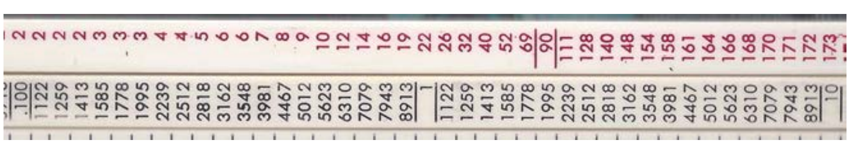
Magnitude, dB; ω = u Red Readings are Negative, Divide Gain readings by 10 for Db



$200\text{Log}1/[1+2\zeta ju/2+(ju/2)^2]$

Frequency u →

Phase Angle θ; ω = u



$\measuredangle 1/[1+2\zeta ju/2+(ju/2)^2]$

Frequency u →

Slide Rule Readings

Frequency ω (u slide rule)	Magnitude, dB	Phase Angle, θ	Comment
0.1 (0.1)	0	-2	
0.5 (0.5012)	+0.5	-9	
1 (1)	+1.9	-22	
1.81 (1.778)	+4.8	-69	$H_{Max}(j\omega)$
2 (1.995)	+4.4	-90	$H(j\omega_0)$
4 (5.012)	-14.8	-164	
10 (10)	-26.6	-173	

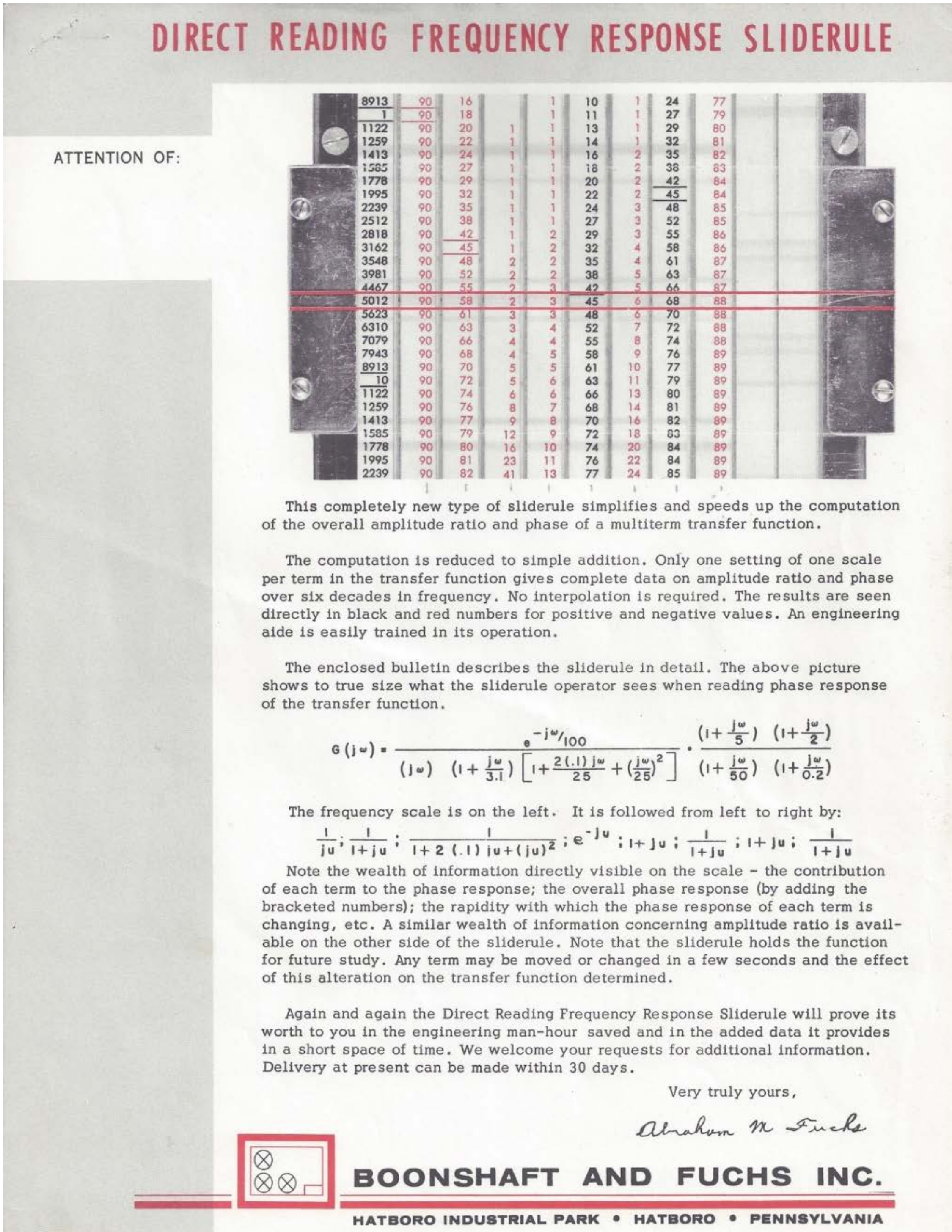


FIGURE 7. Advertisement on the Capabilities of the Boonshaft and Fuchs Slide Rule

Notes

1. JOS Plus indicates that supplemental material for this article is available at www.oughtred.org. There are two files: one contains the instructions and a description of the slide rule, including the Appendix referred to in this article; the other is an explanation of transfer functions as a complex number.