# *On the Calculation of Logarithms – Karl Kommerell's Method of Repeating Squares*

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**Abstract**

Nowadays, the pocket calculator or the Internet provides a quick and easy way to calculate logarithms. Of course, that has not always been the case! In the early days of logarithms, decades were required to calculate all the values that were then published in tables of logarithms.

A brief overview of the history of the computation of logarithms is part of this essay. Newly developed calculation methods essentially always had four objectives:

1. Accuracy.
2. Precision.
3. Speed
4. Simplicity.

To achieve these goals, many mathematicians have set out to develop suitable algorithms that were applied by many calculators - humans or machines - and were used to work out more accurate, more precise, and error-free logarithms. This was not at all about calculating with logarithms but about the calculation of logarithms.

Although in many areas these logarithms were used for calculation purposes, hardly any attention was paid to the calculation methods used to derive these logarithms. Particularly the newer methods, based on the use of series, did not fit into the order of the school curriculum, because series were only discussed after logarithms and to re-visit the old subject of logarithms was not supposed to be appropriate.

However, there were some mathematicians who regretted this fact and have developed methods that help to calculate logarithms without the use of such series, while at the same time improving students' mathematical understanding.

One of those mathematicians was Karl Kommerell (1871 - 1962), a professor in Stuttgart, and his method will be discussed in detail in this paper.

## Introduction

These days, nobody expects anything really new on the subject of logarithms. However, surprisingly there is still something to be written, something to be reviewed, or something to be looked at again with fresh eyes. The history of logarithms has been described and dealt with on many occasions and is very clear. Nevertheless there are unnecessary discussions from time to time about who discovered logarithms. Undeniably, a 7-place logarithmic table (called "*Mirifici Logarithmorum Canonis Descriptio*") was published in 1614 for the very first time by John Napier (1550 – 1617).

Six years later "*Aritmetische und Geometrische Progreß Tabulen / sambt gründlichem Unterricht / wie solche nützlich in allerley Rechnungen zu gebrauchen / und verstanden werden soll*" was published by Jost Bürgi (1552 - 1632) – also known as Joost or Jobst Byrgius or Byrg. This book published logarithms to 8-places; however, the book failed to become popular because there were no instructions for the use of the logarithms. Today, Bürgi is regarded by some as the discoverer of logarithms, because he had been calculating his logarithms years before Napier. However, he only published them on the insistence of Johannes Kepler (1571 – 1630). There is no information about the time John Napier took to finish calculating his logarithms, but in those times the calculations would have taken many years.

After Nicolas Chuquet (1445/1455 – 1487/1488) had documented the principle of a table of logarithms by means of the comparison of tables of arithmetic series and geometric series with a common ratio of 3[2], Michael Stifel (1487 – 1567) in 1544, added negative numerical values to this theory in his *Arithmetica Integra*, using geometric series with a common ratio of 2. Both of these two gentlemen should therefore be regarded as the discoverers of the logarithmic calculation principle. However, the theories of the Babylonians (about 1600 B.C.), Euclid (365 – 300 B.C.), and Archimedes (287 – 212 B.C.) had already begun to move in the logarithmic direction. These theories were based on reducing calculations to a lower level – for example simplifying a multiplication by using an addition.[3]

Following the appearance of the first table of logarithms in 1614, tables containing either the Briggs common (base 10) logarithms or the Napier logarithms quickly became more widespread. The Briggs version proved to be more practical in terms of application and were more often used. This was particularly due to the efforts of the Dutchman Adriaan Vlacq (1600 – 1667), who from 1628 onwards perfected the idea of selling tables of logarithms. The first German edition of the Vlacq tables "*Tabellen der Sinuum, Tangentium und Secantium wie auch der Logarithmorum vor die Sinubus Tangentibus und die Zahlen von 1 bis 10000.*." was published in Amsterdam in 1673 by Joan von Ravesteyn.

However, these tables were not the first ones ever to be published in German. In 1631 Johannes Faulhaber (1580 - 1635) had already published his *"Zehentausend Logarithmi, der Absolut oder ledigen Zahlen von 1 biß auff 10000. Nach Herrn Johannis Neperi Baronis Merchstenij*

*Arth und Invention, welche Heinricus Briggius illustrirt, und Adrianus Vlacq augirt, gerichtet"* (re-edited 1637) as a supplement to his *Ingenieurs Schul* dated 1630.

Faulhaber himself said in this publication "*Aus diesem erklärten Bericht kann man nun mehr verstehen den 'Ursprung oder Geburt der Logarithmorum, welches ein solche herrliche schöne Kunst / so mit Worten nicht auszureden ist. / Dann ist es nicht ein wunderbarliche Invention, daß in währendem Proceß das Extrahieren ins Halbieren verwandelt wird ? Und man die Zahlen ausrechnen und erforschen kan / daß ihre heimlichen Naturen offenbar werden müssen ? Wie ich dann zum Beschluß noch diß Exempel setzen will…....*".

Rough translation: That report stated, you can now understand the 'origin or birth of Logarithms which / is such a wonderful beautiful art, not to be describable with words. / Then isn't it a peculiar invention that the extraction of square roots is transformed into halving? And you can figure out and explore the numbers / that their secret natures will be revealed? Finally, I will formulate this example…

This work can be described as the first German mathematical textbook containing a table of logarithms and also including logarithms in the calculations with numerous examples taken from all areas of the art of numeration at that time.

Then, in 1618 Benjaminus Ursinus (1587 - 1633/4) published in Coloniae/Kölln near Berlin (now a part of Berlin itself) the first table of logarithms published on German/continental European ground. This table was written in Latin, and contained the Napier logarithms reduced to 5 places.

Karl Röttel has shown in a diagram (see Figure 1) those topics in which calculation with logarithms were and are still in use. But how have the logarithms in this and the following tables been calculated?

## General Background on the History of Calculating Tables of Logarithms

The history of the calculation of logarithms is closely linked to the history of tables of logarithms – at least in the early stages from 1614 to 1660. Every issuer of tables had found his own way of calculating the logarithms in the tables he published.

The people calculating logarithms had to establish a meaningful relationship between the values of geometric and arithmetic series, and thereby to determine the logarithm value of a number (the numerus). In order to cover as many numbers (numeri) as possible, the distances between two adjacent numbers were kept to a minimum. Negative logarithm values were still avoided.

While the basis (common ratio of the geometric series) values can be calculated for Bürgi and Napier logarithms, neither of them consciously used a specific basis. Instead, they approached the calculation of values intuitively. Bürgi using the "antilogarithmic" method[4] and Napier the "kinematic" method[5]. Both of these methods are extensively described in the appropriate literature.

What follows is a tabular summary of the first tables of logarithms published by the calculators and authors in the 17th century:

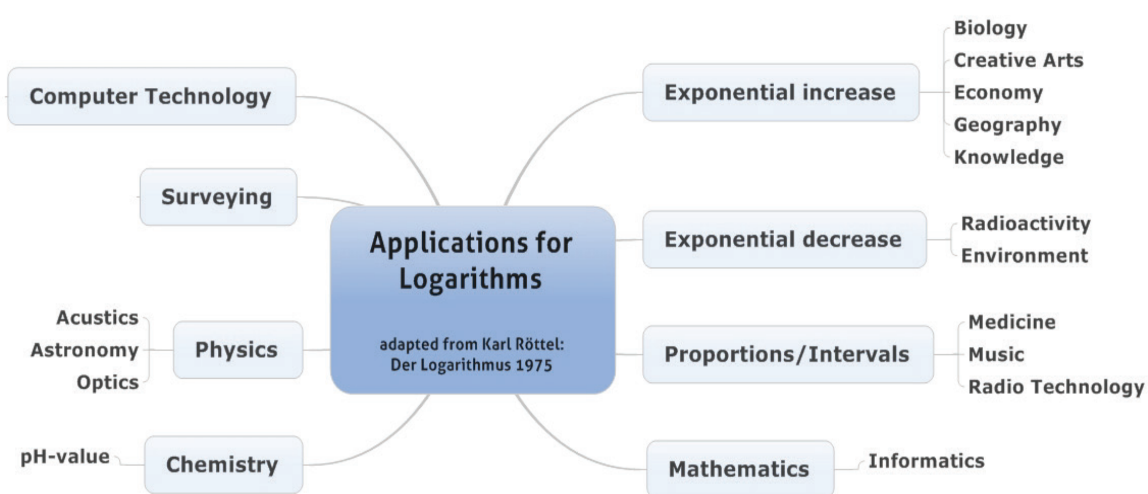


**FIGURE 1. Topics Using Logarithm Calculations**

**TABLE 1. 17th Century Logarithm Tables**

| Year | Author | Title |
|---|---|---|
| 1614-1619 | John Napier (1550 – 1617) | *Mirifici Logarithmorum Canonis Descriptio/ - Constructio* |
| 1617 | Henry Briggs (1561 – 1630) | *Logarithmorum Chilias Prima* |
| 1619 | Euclid Speidell (1600 - 1634) | *New logarithmes the first invention whereof, was, by the honourable Lo. Iohn Nepair, Baron of Marchiston, and printed at Edinburg in Scotland, anno 1614, in whose vse was and is required the knowledge of algebraicall addition and substraction, according to + and - : these being extracted from and out of them (they being first ouer seene, corrected, and amended) require not at all any skill in algebra, or cossike numbers, but may be vsed by euery one that can onely adde and substract ... / by Iohn [Spe]idell ... , [London] : ... to bee [solde at his] dwelling house in the Fields ..., (remark: first log nat of N)* |
| 1620 | Jost Bürgi (1552 – 1632) | *Aritmetische und Geometrische Progreß Tabulen* |
| 1620 | Edmund Gunter (1581 - 1626) | *Canon triangulorum, sive tabulæ sinuum et tangentium artificialium* |
| 1624 | Johannes Kepler (1571 - 1630) | *Mathematici chilias logarithmorum ad totidem numeros rotundas* |
| 1625 | Edmund Wingate (1596-1656) | *Arithmetique logarithmique, or La Construction & Usage des Tables Logarithmetiques* |
| 1633 | Nathaniel Roe (1596-1656) | *Tabulæ logarithmicæ, or two tables of logarithmes* |
| 1668 | Nicolaus Mercator (1620 - 1687) | *Logarithmotechnia: sive methodus construendi logarithmos nova, accurata, & facilis* |

In recent times Denis Roegel[6] has rendered outstanding services to the description, explanation, renovation (reconstruction), and maintenance of more ancient and significant tables of logarithms.

The calculation methods applied for the logarithms were subjected to continuous updating. Methods in later centuries took advantage of the progress made in mathematics, which resulted in the development of mathematical series and their appropriate application in the calculation of logarithms.

A general summary of the history of the calculation methods, often linked to the history of logarithms, can be found in the following works:

**TABLE 2. History of Calculation Methods**

| Author | Title | Year | Reference |
|---|---|---|---|
| Benjamin Martin | *Logarithmologica* | 1740 | 7 |
| Henry Sherwin | *Sherwin's mathematical tables* | 1742 | 8 |
| Charles Hutton | *Mathematical Tables* | 1822 | 9 |
| Florian Cajori | *A History of Mathematics* | 1919 | 10 |
| Charles Naux | *Histoire des Logarithmes* | Bd. 1, 1966; Bd. 2 1971 | 11 |

However, expanding on all these mathematical theories would go far beyond the scope of this essay. Furthermore, in recent times these methods of calculation have been explained in many publications. In Gerlinde Faustmann's doctoral thesis[12] there is a good compilation of the history of the calculation methods for logarithms.

### Kommerell's Method

The method[13] described by Karl Kommerell for calculating logarithms by repeated squaring is described as follows and is entirely from his publication dated 1917, since his description is an original text, which is clear and comprehensible. To begin the explanation, Kommerell uses a method involving calculation with partial fractions, by which every number can be expressed as shown below. This particular case is a dyadic fraction as the denominators are integer powers of 2.

$$x = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_1} * 2^{\alpha_2}} + \frac{1}{2^{\alpha_1} * 2^{\alpha_2} * 2^{\alpha_3}} + \ldots\ldots \quad (1)$$

Here $\alpha_1, \alpha_2, \alpha_3, \ldots$ are positive whole numbers

Kommerell writes:

If equation (1) is multiplied by $2^{\alpha_1}$, the result is:

$$x * 2^{\alpha_1} = 1 + \frac{1}{2^{\alpha_2}} + \frac{1}{2^{\alpha_2} * 2^{\alpha_3}} +$$

$\alpha_1$ is therefore the number which indicates how often x has to be multiplied by 2, so that $x * 2^{\alpha_1}$ just exceeds unity. If 1 is subtracted from $x * 2^{\alpha_1}$, the result is:

$$x * 2^{\alpha_1} - 1 = \frac{1}{2^{\alpha_2}} + \frac{1}{2^{\alpha_2} * 2^{\alpha_3}} + \ldots\ldots$$

$x * 2^{\alpha_1} - 1$

is now multiplied by 2 until unity – again – is just exceeded; $\alpha_2$ results from the number of these multiplications etc. Now, consider some number b, which it can be assumed lies between 1 and 10; obtaining the base 10 logarithm of this number requires that the equation be solved, and we know that x lies between 0 and 1.

$$10^x = b \qquad (2)$$

To solve (2), consider x to be the dyadic fraction shown above as (1) giving the equation:

$$10^{\frac{1}{2^{\alpha_1}}+\frac{1}{2^{\alpha_1}*2^{\alpha_2}}+\frac{1}{2^{\alpha_1}*2^{\alpha_2}*2^{\alpha_3}}\ldots\ldots+\frac{1+\rho}{2^{\alpha_1}*2^{\alpha_2}\ldots\ldots2^{\alpha_k}}} = b \qquad (3)$$

in which $\rho$ is a positive number < 1. The numbers $\alpha_1, \alpha_2, \alpha_3, \ldots \rho$ here are still to be defined. Raising equation 3 with the exponent $2^{\alpha_1}$, one gets

$$10^{1+\frac{1}{2^{\alpha_2}}+\frac{1}{2^{\alpha_2}*2^{\alpha_3}}+\frac{1}{2^{\alpha_2}*2^{\alpha_3}*2^{\alpha_4}}\ldots\ldots+\frac{1+\rho}{2^{\alpha_2}*2^{\alpha_3}\ldots\ldots2^{\alpha_k}}} = b^{2^{\alpha_1}}$$

This identity shows that $\boldsymbol{\alpha_1}$ defines how often $\boldsymbol{b}$ has to be squared, so that 10 < $\boldsymbol{b^{2^{\alpha_1}}}$ < 100. The last equation is divided by 10, giving the result:

$$10^{\frac{1}{2^{\alpha_2}}+\frac{1}{2^{\alpha_2}*2^{\alpha_3}}\ldots\ldots+\frac{1+\rho}{2^{\alpha_2}*2^{\alpha_3}\ldots\ldots2^{\alpha_k}}} = b^{2^{\alpha_1}} * 10^{-1} \qquad (4)$$

with $1 < b^{2^{\alpha_1}} * 10^{-1} < 10$. $\alpha_2$ is handled in the same way as was previously $\alpha_1$. $b^{2^{\alpha_1}} * 10^{-1}$ has to be squared as many times until 10 is just exceeded; the number of operations necessary is $\alpha_2$. In this way $\alpha_1$ can be defined very quickly, and there follows:

$$x = log\, b = \frac{1}{2^{\alpha_1}} + \frac{1}{2^{\alpha_1}*2^{\alpha_2}} + \ldots \frac{1+\rho}{2^{\alpha_1}*2^{\alpha_2}\ldots 2^{\alpha_k}} \qquad (5)$$

In which only the number $\rho$ between 0 and 1 is unknown. If in (5) the number $\rho$ is omitted, the resulting error is smaller than

$$\frac{1}{2^{\alpha_1}*2^{\alpha_2}\ldots 2^{\alpha_k}} \qquad (6)$$

As an example, because

$$\frac{1}{2^{20}} < 0.000\,001 \qquad (7)$$

One may calculate the 5 digit logarithm of $b$ by following the squaring process 20 times. If the ratios are: $\frac{1}{2^1}$ , $\frac{1}{2^2}$ … $\frac{1}{2^{20}}$ , one is only required to add those numbers from equation (5).

**To Calculate the Logarithm of 1.4562**

In Table 3, the first number in each row has been squared as many times as are necessary in order to exceed the number 10, and that number is then divided by 10 and used as the first number in the next row. The column headings show the number of times, the $\alpha$, that the number has to be squared to exceed 10. In this table, the first five squarings are calculated precisely to the fifth decimal place, the next five squarings precisely to the fourth decimal place, and so on. As can be seen in Table 3, the logarithm can be safely calculated precisely to five decimal places.

**TABLE 3. Calculating Logarithms by Squaring**

| Squaring process. | 1 | 2 | 3 | 4 | $\alpha$ | Calculation |
|---|---|---|---|---|---|---|
| **1.4562** | 2.12052 | 4.49660 | 20.21941 | | $\alpha_1 = 3$ | $\frac{1}{2^3} = 0.125\,000$ |
| 2.02194 | 4.08824 | 16.71371 | | | $\alpha_2 = 2$ | $\frac{1}{2^5} = 0.031\,2500$ |
| 1.6717 | 2.7936 | 7.8042 | 60.9055 | | $\alpha_3 = 3$ | $\frac{1}{2^8} = 0.003\,9063$ |
| 6.09055 | 37.0948 | | | | $\alpha_4 = 1$ | $\frac{1}{2^9} = 0.001\,9531$ |
| 3.7095 | 13.7604 | | | | $\alpha_5 = 1$ | $\frac{1}{2^{10}} = 0.000\,9765$ |
| 1.376 | 1.893 | 3.583 | 12.838 | | $\alpha_6 = 3$ | $\frac{1}{2^{13}} = 0.000\,1220$ |
| 1.284 | 1.649 | 2.719 | 7.39 | 54.61 | $\alpha_7 = 4$ | $\frac{1}{2^{17}} = 0.000\,0076$ |
| 5.46 | 29.81 | | | | $\alpha_8 = 1$ | $\frac{1}{2^{18}} = 0.000\,0038$ |
| 2.98 | 8.88 | 78.85 | | | $\alpha_9 = 2$ | $\frac{1}{2^{20}} = 0.000\,0009$ |
| **log 1.4562 =** | | | | **Total** | | **0.163 2202** |

**TABLE 4. The Kommerell Method to Calculate a Logarithm**

| | 1 | 2 | 3 | 4 | 5 | 6 | $\alpha$ | n | Calculation* |
|---|---|---|---|---|---|---|---|---|---|
| **3.5258** | 12.431266 | | | | Go to next row when squares are > 10 and divide by 10 | | $\alpha_1 = 1$ | $1 = \alpha_1$ | 0.5000000 |
| 1.24312 | 1.5453473 | 2.3880984 | 5.7030139 | 32.524367 | | | $\alpha_2 = 4$ | $5 = \alpha_1 + \alpha_2$ | 0.0312500 |
| 3.254 | 10.588516 | | | | | | $\alpha_3 = 1$ | $6 = \alpha_1 + \alpha_2 + \alpha_3$ | 0.0156250 |
| 1.0588 | 1.1210574 | 1.2567698 | 1.5794703 | 2.4947264 | 6.2236598 | 38.7339411 | $\alpha_4 = 6$ | 12 | 0.0002441 |
| 3.873 | 15.000129 | | | | | | $\alpha_5 = 1$ | 13 | 0.0001221 |
| 1.50001 | 2.25003 | 5.062635 | 25.630273 | | | | $\alpha_6 = 3$ | 16 | 0.0000153 |
| 2.563027 | 6.5691074 | 43.153172 | | | | | $\alpha_7 = 2$ | 18 | 0.0000038 |
| 4.3153 | 18.621814 | | | | | | $\alpha_8 = 1$ | 19 | 0.0000019 |
| 1.8621 | 3.4674164 | 12.022977 | | | | | $\alpha_9 = 2$ | 21 | 0.0000005 |
| | | | **log 3.5258 =** | | | | | **Total** | **0.5472627** |

*Calculation: $\frac{1}{2^n} = \frac{1}{2^1} = 0.500\,000$ $\frac{1}{2^n} = \frac{1}{2^{18}} = 0.000\,0038$

The addition of the numbers in the last column of Table 1 is:

***log 1.4562 = 0.163 2202***

which is exactly as determined in the table of logarithms. The diagram contains the whole calculating operation. No auxiliary calculation has been omitted. If the logarithmic series were to be used to calculate log 1.4562 the calculations would have to be much longer and much more unpleasant. However, for calculating the logarithms of all numbers, the logarithmic series again is applicable, because, as is generally known, the logarithmic series can be transformed in different ways, so that only a few members of the series are needed for the calculation.

Performing these calculation steps eases the understanding of Kommerell's words:

> It is even conceivable that, by using this method, a machine could be built which directly provides the logarithm for every number.

If the sequence of the operations calculated in Table 1 is reversed, the result is the antilogarithm of each of the given logarithms.

Kommerell continues by explaining how, by using square roots, to work backwards from a logarithm to an antilogarithm.

Table 4 is an example, using an Excel spreadsheet, showing how to calculate **log 3.5258** using the Kommerell method.

Logarithms calculated by his method are not known to have been used in tables of logarithms.

Logarithms celebrated their 400th birthday in Edinburgh in 2014. See the separate report in this issue.

**TABLE 5. The Life of Karl Kommerell (1871 – 1962)**[14]

| | | |
|---|---|---|
| 19. 8.1871 | | Born in Achern (Baden) |
| ???? | | Gymnasium Tübingen (secondary school) |
| 1890 -1895 | | Studied mathematics at Tübingen University |
| 1897-1898 | | Studied in Paris |
| 1897 | | Graduation to Dr. rer. nat. (PhD) Tübingen University |
| 1899 | | Married Anna Stammler from Tübingen |
| 1900 | | Senior preceptor in Ravensburg |
| 1907 | | Secondary school professor in Heilbronn |
| 1908 | | Professor at Friedrich-Eugens-Realschule in Stuttgart |
| 1911 | | State doctorate University of Technology Stuttgart |
| 1921 | | Associate professor University of Technology Stuttgart |
| 1925 | | Full professor of mathematics Tübingen University |
| 1936 | | Retired - Professor Emeritus |
| 30. 7. 1962 | | Died in Tübingen |

**TABLE 6. Publication List of Karl Kommerell (selection)**

- *Die Krümmung der zweidimensionalen Gebilde im ebenen Raum von vier Dimensionen.* Dissertation Tübingen 1897, 53 S.
- *Zur Auflösung algebraischer Gleichungen. Math. nat. Mitt. Württ.* (2) 1 (1899), S.15-23
- *Die nichteuklidische Geometrie und Trigonometrie auf den Flächen von konstantem Krümmungsmaß. Math. nat. Mitt. Württ.* (2) 3 (1901), 5.18-31
- *Rein geometrische Begründung der Lehre von den Proportionen und des Flächeninhalts.* Math. Ann. 66 (1909), S.558
- *Über die Anschauung in der Mathematik, Akademische Antrittsvorlesung. Math. nat. Mitt. Württ.* (2) 14 (1912), S. 1
- *Der Sylvestersche Plagiograph und die Proportionenlehre. Jber. DMV* 21 (1912), 5.173-176
- *Elementare Berechnung der Zahl pi und die trigonometrischen Funktionen. Arch. Math. Phys.* 25(1917) 14S.
- *Berechnung der Logarithmen durch wiederholtes Quadrieren. Arch. Math. Phys.* 28 (1919/20), 9 S.
- *Affine Raumtransformationen und Affinoren. Jber. DMV* 3o (1921), 5.35-55
- *Vektorielle Begründung der sphärischen Trigonometrie. Jber. DMV* 32 (1923), S.86-gi
- *Über den Pohlkeschen Satz und die Konstruktion von Trägheitsellipsen. ZAMM 8* (1928), S. 7o73
- *Das System der Schraubenachsen bei beliebigen Bewegungen. Jber. DMV* 38 (1929), 5.281-284
- *Das Grenzgebiet der elementaren und höheren Mathematik in ausgewählten Kapiteln dargestellt.* Leipzig 1936
- *Der Desarguessche Involutionssatz. Jber. DMV* 52 (1942), 5.55-56 hinten
- *Grundlagen der Euklidischen und Nicht-Euklidischen Geometrie.* 1949

Further articles on the subject of logarithms including documentation of more than 3000 tables of logarithms can be found in *Collectanea de Logarithmis*, a DVD for purchase, edited and published by the author of this paper.[15]

## Background Information on Partial Fraction and Partialising Calculation

Having enquired among fellow mathematicians, the expression calculation by partial fractions is not very familiar. For this reason further information on the subject is given below; this information is taken unabridged from the literature quoted.

The works of Dr. Eduard Heis, mathematician and astronomer (1806 – 1877)[16], mark the starting point of calculation by partial fractions. These are described in considerable detail in his "*Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra*"[17] and in "*Schlüssel zur Sammlung….*"[18] in § 86 *“series for partial fractions”*. Heis writes:

> A special type of approximation for multi-digit simple fractions or decimals which is of practical application, is obtained if one converts such a fraction into a series of fractions, all of which have numerator 1, and of which each of the members is an aliquot part (that is, a whole divisor) of the immediately preceding member, i.e. a series of the form:

$$\frac{1}{x}+\frac{1}{x*y}+\frac{1}{x*y*z}+\frac{1}{x*y*z*u}+\frac{1}{x*y*z*u*v}+\cdots,$$

or if the first fraction is $A_1$, the second $A_2$, the third $A_3$ etc., one gets the form:

$$\frac{1}{x}+\frac{1}{y}A_1+\frac{1}{z}A_2+\frac{1}{u}A_3+\frac{1}{v}A_4+\cdots$$

Heis calls such successive fractions “partial fractions” and the series “partial fraction series”. They were first applied to illustrate common fractions, square and cubic roots, logarithms, and the roots of equations. The ancient Egyptians used partial fractions in practical mathematics.

If this series is stopped at any one point, an approximation results, which gets closer to the real value the more fractions are added.

One may represent this series as a finite ascending continued fraction:

$$\cfrac{1+\cfrac{1+\cfrac{1+\frac{1}{u}}{z}}{y}}{x}$$

Such fractions differ from the more commonly encountered descending continued fraction in that, in the ascending variant, the numerator progresses in a way analogous to the denominator of the descending variant.

An example of a calculation from E. Heis follows[19]:

Express $\frac{1301}{5720}$ as a series of partial fractions, (with the aim of achieving an approximation to the desired precision).

For example, $\frac{1301}{5720}=\frac{1}{x}+\frac{1}{x*y}+\frac{1}{x*y*z}+\frac{1}{x*y*z*u}+\cdots$

or

$$1301\,x = 5720 + \frac{5720}{y} + \frac{5720}{y*z} + \frac{5720}{y*z*u} + \cdots$$

5720, when divided by 1301, gives a non-integer quotient, which is between 4 and 5; however x, like y, z, etc., must be an integer; therefore we use x = 1 + 4 = 5; there then follows:

$$6505 = 5720 + \frac{5720}{y} + \cdots \mathit{and\ from\ here}$$

$$785y = 5720 + \frac{5720}{z} + \frac{5720}{z*u} + \cdots \quad \mathit{with}\ y = 8$$

$$560\,z = 5720 + \frac{5720}{u} + \cdots \qquad \mathit{with}\ z = 11$$

$$440\,u = 5720 + \cdots \qquad \mathit{with}\ u = 13$$

The result is therefore:

$$\frac{1301}{5720} = \frac{1}{5} + \frac{1}{5*8} + \frac{1}{5*8*11} + \frac{1}{5*8*11*13}$$

$$\frac{1301}{5720} = \frac{1}{5} + \frac{1}{8}A_1 + \frac{1}{11}A_2 + \frac{1}{13}A_3$$

The approximations are: $\frac{1}{5}; \frac{9}{40}; \quad \frac{100}{440} = \frac{5}{22}$

When using the method of continued fractions, the resultant approximations are:

$$\frac{1}{4}, \frac{2}{9}, \frac{3}{13}, \frac{5}{22}, \frac{58}{253}, \frac{237}{1042}, \frac{532}{2339}.$$

For faster calculation of the parts 5, 8, 11, and 13, one may use the following scheme:

**TABLE 7. Calculation Scheme**

| | | | |
|---|---|---|---|
| | | 5720/1301 | = 5 = *x* |
| | | 6505 | |
| | 5720 / | 785 | = 8 = *y* |
| | 6280 | | |
| | 5720/ 560 | | = 11 = *z* |
| | 6160 | | |
| 5720 / | 440 | | = 13 = *u* |

## Partialising Calculation

Streets in Germany have been named after Eduard Heis in Cologne-Flittard and in Münster as a tribute to the mathematician and astronomer. One of his important "discoveries" concerned ways to apply partial fractional series. Kommerell used these methods later to calculate logarithms. How he did that is described in the main article above.

As E. Heis mentioned, the Egyptians knew about, and applied, partial fractions. One example originates from AHMES[20], who was a scribe who copied, but probably did not write, the Rhind Papyrus around 1700 B.C.**:**

$$\frac{2}{17} = \frac{1}{12} + \frac{1}{51} + \frac{1}{68}$$

This shows a decomposition into "unit fractions", each having the numerator 1 and the smallest value at the end. In 1911 Heinrich Wieleitner said: "No more is known about this method of calculation[21]". This is no longer the case, as Karl Kleine has kindly informed me.[22]

Subsequently, some older arithmetic books have been examined. As an example 19th Century arithmetic books refer to *Vermischungs*-, *Partitions- oder Gesellschafts-rechnungen* in connection with partialising calculations, and these are often dealt with in the chapter after continuous fractions. Here is an example exercise taken from Poppe:[23]

```
4 merchants pooled money into a common
trade as follows: A 1200 fl, B 1800 fl, C
1000 fl, und D 1400 fl. In doing so they
made a profit of 2000 fl. How much is the
share of each individual merchant in this
profit?
```

Vega[24] defines a partialising calculation as follows:

```
If a whole is to be divided into several
unequal parts which all have to be related
to each other, the way of doing this is
called a partialising calculation or
partialising rules.
```

This presumably somewhat older partialising calculation method (*Teilrechnung*, described as *Proportionale Verteilung*, meaning "proportional distribution") does not have a lot to do with partial fraction calculation **(***Teilbruchrechnng*), although their German terms are very similar. The partialising calculation cannot be applied in the calculation of logarithms.

## Conclusion

This article has described one of the latest (pre-computer) methods of manually calculating logarithms and antilogarithms.

## Notes


1. First presented and published in German at the IM 2013 in Berlin. Jerry McCarthy´s help in putting this updated article into correct English is gratefully acknowledged.
2. Erwin Voellmy: *Jost Bürgi und die Logarithmen*; Birkhäuser Verlag Basel, 2. Auflage, 1974.
3. Gerlinde Faustmann: *Österreichische Mathematiker um 1800 unter besonderer Berücksichtigung ihrer logarithmischen Werke*; Dissertation, Vienna, 1992.
4. Heinz Lutstorf, Max Walter: *Jost Bürgi´s Progress Tabulen*; publication series of ETH-Library, Zürich, 1992.
5. Joachim Fischer: *Ein Blick vor bzw. hinter den Rechenschieber: Wie konstruierte Napier Logarithmen?* 1997; http://rechnerlexikon.de/files/LogNapier-F.pdf
6. Denis Roegel: http://locomat.loria.fr/
7. Benjamin Martin: *Logarithmologica: or the whole doctrine of Logarithms, common and logistical in Theory and Practice - in three parts*; J. Hodges, London, 1740.
8. Henry Sherwin: *Sherwin's mathematical tables, contriv'd after a most comprehensive method: containing Dr. Wallis's account of logarithms, Dr. Halley's and Mr. Sharp's ways of constructing them; with Dr. Newton's contraction of Brigg's logarithms, viz. A table of logarithms of the numbers from 1 to 101000, with the means to find readily the logarithm of any number, and the number of any logarithm, to seven places of figures: And tables of the natural and logarithmic sines, tangents, secants, and versed-sines, to every minute of the quadrant: with the explication and use prefix'd*. (Third Edition by William Gardiner); William Mount and Thomas Page London, 1742.
9. Charles Hutton: *Mathematical Tables: containing Common, Hyperbolic, and Logistic Logarithms…*; Rivington, London, 1822.
10. Florian Cajori: *A History of Mathematics*; The Macmillan Company New York. 1919.
11. Charles Naux: *Histoire des Logarithmes – de Neper a Euler*; Librairie Scientifique et Technique A. Blanchard; Paris 1966/1971.
12. Gerlinde Faustmann: ibid.
13. Karl Kommerell: *Die Berechnung der Logarithmen durch wiederholtes Quadrieren*; Arch. Math. Phys. 28, pages 137 – 145 (1919/1920).
14. Karl-Heinz Böttcher, Bertram Maurer: *Stuttgarter Mathematiker – Geschichte der Mathematik an der Universität Stuttgart von 1829 bis 1945 in Biographien*; Universität Stuttgart, 2008.
15. Dr. Klaus Kühn, napier2014@iasim.de will be grateful for any errors pointed out and for any supplementary information.
16. Cf. http://en.wikipedia.org/wiki/Eduard_Heis
17. Eduard Heis: *Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra 106. – 108. sehr verbesserte Auflage*, Verlag der M.DuMont-Schaubergischen Buchhandlung, Köln, 1904.
18. Eduard Heis: *Schlüssel zur Sammlung von Beispielen und Aufgaben aus der allgemeinen Arithmetik und Algebra*, 2. verbesserte Auflage, Verlag der M. DuMont-Schaubergischen Buchhandlung, Köln, 1878.
19. A. Kleyer; *Lehrbuch der arithmetischen und geometrischen Progressionen der zusammengesetzten, harmonischen, Ketten- und Teilbruchreihen*, Verlag L. v. Vangerow, Bremerhaven, 1884.
20. Fritz Müller: *Im Anfang war die Zahl*; Europäischer Buchklub, Stuttgart Zürich Salzburg, 1957; cf. http://en.wikipedia.org/wiki/Rhind_Mathematical_Papyrus
21. Heinrich Wieleitner: *Der Begriff der Zahl*; B.G. Teubner, Leipzig, Berlin, 1911.
22. http://en.wikipedia.org/wiki/Egyptian_fraction
23. Johann Heinrich Moritz Poppe: *Die Volks-Größenlehre*, Metzlerische Buchhandlung, Stuttgart, 1827.
24. Georg Freiherrn von Vega: *Vorlesungen über die Mathematik*, first volume 7th. Edition revised by Wilhelm Matzka, Fr. Beck´s Universitätsbuchhandlung, Wien, 1850.