Design Considerations for the Analon Slide Rule

Cliff Frohlich

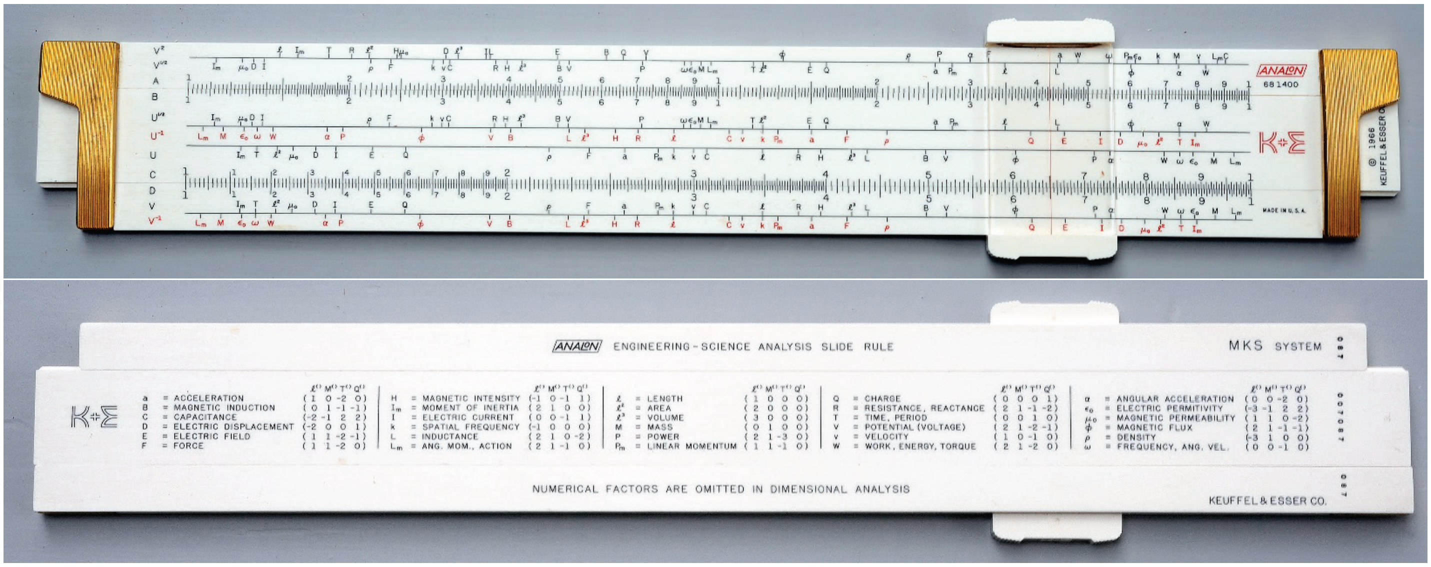

FIGURE 1. The Analon Slide Rule
Top - Front; Bottom - Back

Introduction

The Analon slide rule (see Figure 1) has achieved iconic status among slide rule enthusiasts. Keuffel & Esser (K&E) patented the Analon in 1966, and their 1967 catalog[1] touted it as "a revolutionary concept in slide rules" because "it allows the student, professional engineer and scientist to perform reliable dimensional calculations". However, manufacturing problems delayed production, and, when electronic hand calculators became available in 1971-72, the market for the slide rules virtually disappeared. As a result the Analon is moderately rare; fewer than 600 were released; many slide rule collectors have never handled one. The layout is relatively attractive as slide rules go. The Analon is occasionally mentioned as a candidate for "the holy grail of slide rules" on collectors' blogs that discuss such issues.[2] The blogs have also questioned whether or not dimensional analysis slide rules would have caught on and achieved widespread acceptance and use, if slide rules had not been displaced by hand calculators.

The literature on the Analon is not vast, and there is some confusion about the inventor. Joseph Soper[3], who was a K&E employee while the Analon was manufactured, states:

> `Michael P. Smyth, Ph.D., Assistant Professor of Engineering, University of Pennsylvania, designed this unusual rule.`

However, Alfred B. Pikus is the sole author of the Analon patent[4], which does not mention Smyth. Smyth is the sole author of the Analon manual[5], which never mentions Pikus. Yet Pikus is a coauthor with Smyth on an article in an engineering journal explaining the concepts behind the Analon[6]. Quite frankly, Pikus' patent provides a far more lucid explanation of these concepts than does the Smyth and Pikus article. I speculate that Pikus was the inventor responsible for the Analon concept, but that Smyth consulted for K&E during the design and production of the Analon rule.

This article will focus on explaining why the Analon works; i.e., exploring the mathematical principles embodied in its design that allow it to perform dimensional analysis. The Analon's U and V dimensional analysis scales are its most innovative feature. Although the Analon did include the conventional C, D, A, and B scales for computation, these were supplied almost as if by afterthought — the 32-page Analon manual[5] provides no explanation how to use them, stating only that:

> `If you are not familiar with… a slide rule, refer to one of the many manuals on the subject.`

In this article I first review briefly how to do dimensional analysis without a slide rule (the conventional way), and with the Analon (see also the article by O'Leary[7]). Then I discuss why the Analon works and propose some workable alternative designs. I evaluate these designs to gain insight into the design priorities employed by K&E engineers when they designed the Analon. Finally I return to the question of whether the Analon design might have achieved widespread acceptance.

Dimensional Analysis

Dimensional analysis is familiar to all physics and engineering students. Dimensional analysis is based on the observation that while the numerical value for a physical quantity such as energy depends on the system of units (i.e., energy units are joules in MKS units or foot-pounds force in

US/imperial units), energy or any other physical quantity can be expressed as a unique combinations of four fundamental quantities: length, time, mass, and (electric) charge.[8]

For example, we can express energy in terms of fundamental quantities as:

$$\text{(energy)} = \text{(mass)} \cdot \text{(length)}^2/\text{(time)}^2 \qquad (1)$$

or equivalently, we can simply list the values of the exponents n_L, n_M, n_T, and n_Q of length, mass, time, and charge:

$$\text{energy: } [n_L, n_M, n_T, n_Q] = [2,1,-2,0]. \qquad (2)$$

Suppose that one day, Albert Einstein wanted to calculate the energy associated with the rest mass of the electron, but he could not remember whether his now-famous formula was:

$$E = mc^2 \text{ or } E = mc^3$$

Thus, before determining the numerical answer on his slide rule (a Nestler, we are told), he undoubtedly would have checked that:

$$\text{kg} \cdot \text{m/sec} \cdot \text{m/sec} = \text{kg} \cdot \text{m}^2/\text{sec}^2$$

which is the MKS unit for energy in joules, whereas kg · m^3/sec^3 is not. Equivalently, he could have compared:

	$[n_L, n_M, n_T, n_Q]$
mass:	[0, 1, 0, 0]
+ velocity:	[1, 0, -1, 0]
+ velocity:	[1, 0, -1, 0]
energy:	[2, 1, -2, 0]

and:

	$[n_L, n_M, n_T, n_Q]$
mass:	[0, 1, 0, 0]
+ velocity:	[1, 0, -1, 0]
+ velocity:	[1, 0, -1, 0]
+ velocity:	[1, 0, -1, 0]
NOT energy:	[3, 1,- 3, 0]

Had Einstein owned an Analon, he would have checked mc^2 by using the U and V scales to multiply M · v · v (M and lower case v being the locations on the Analon for mass and velocity, respectively; (see Figure 2) confirming that the cursor fell on W (for work or energy). To check mc^3 he would have multiplied by v again, placing the cursor in an unmarked location between F (force) and a (acceleration) on the V scale, nowhere near W.

Possible Design Principles for the Analon

How does the Analon design allow this calculation to work? Although the Analon adds and subtracts distances along the U and V scales to accomplish multiplication and division of physical quantities like mass and velocity, the Analon's design has nothing to do with logarithms. Rather, the design depends on some simple facts about linear combinations like

$$X = n_L D_L + n_M D_M + n_T D_T + n_Q D_Q \qquad (3)$$

where n_L, n_M, n_T, and n_Q are positive or negative integers or zero, and the four D's are basis lengths; i.e., fixed lengths chosen in an arbitrary but convenient way to represent powers of length, mass, time, and charge.

Thus, in our calculation for mc^2 above, moving the slide and cursor around allowed us to identify the location marked by W where

$$X_W = 2D_L + D_M - 2D_T \qquad (4)$$

or $X_W = 19.18$ for the values of D_L, D_M, D_T, and D_Q measured for the Analon (see Table 1). However, as with a conventional slide rule, because the Analon is only 10 inches long[9], the rule ignores multiples of 10 and finds the location of W to be 9.18, dropping the ten[10].

Although the basis lengths D_L, D_M, D_T, and D_Q are chosen arbitrarily by the slide-rule designer, not just any values will do; they must be chosen so that the values of equation (3) are distinct for all 30 physical quantities of interest. For example, the following design would not work: suppose we chose the values for D_L and D_T to be 4 units and 2 units, respectively. Then the locations X_T and X_v for time T and velocity v would both be 2 units (since $X_v = D_L - D_T$). This leads to:

<u>Design Principle #1: The values of equation (3) must be different for all physical quantities of interest</u>.

The Analon incorporates 30 physical quantities, as listed on the back of the rule (see Figure 1 and Table 2).

Not only must the lengths D_L, D_M, D_T, and D_Q be chosen to make the 30 values of equation (3) distinct, we would prefer that none of the 30 values are too close together. That is, we want to avoid finding the wrong answer as errors accumulate when we perform a dimensional analysis with our rule. For example on the Analon, the closest-together values (see Figure 3) are between ω and ε_o, and between v and C, both separated by about 0.12 units distance. This suggests:

<u>Design Principle #2: For all physical quantities of interest, the minimum separation S_{MIN} for values of equation (3) should be as large as possible.</u>

Moreover, we want to choose D_L, D_M, D_T, and D_Q to use the whole length of the slide rule. We do not want all 30 values for equation (3) on one end of the rule; we don't want big blank spaces anywhere with no values for equation (3). Thus, another design principle is:

<u>Design Principle #3: For all physical quantities of interest, the maximum separation S_{MAX} for values of equation (3) should be as small as possible</u>.

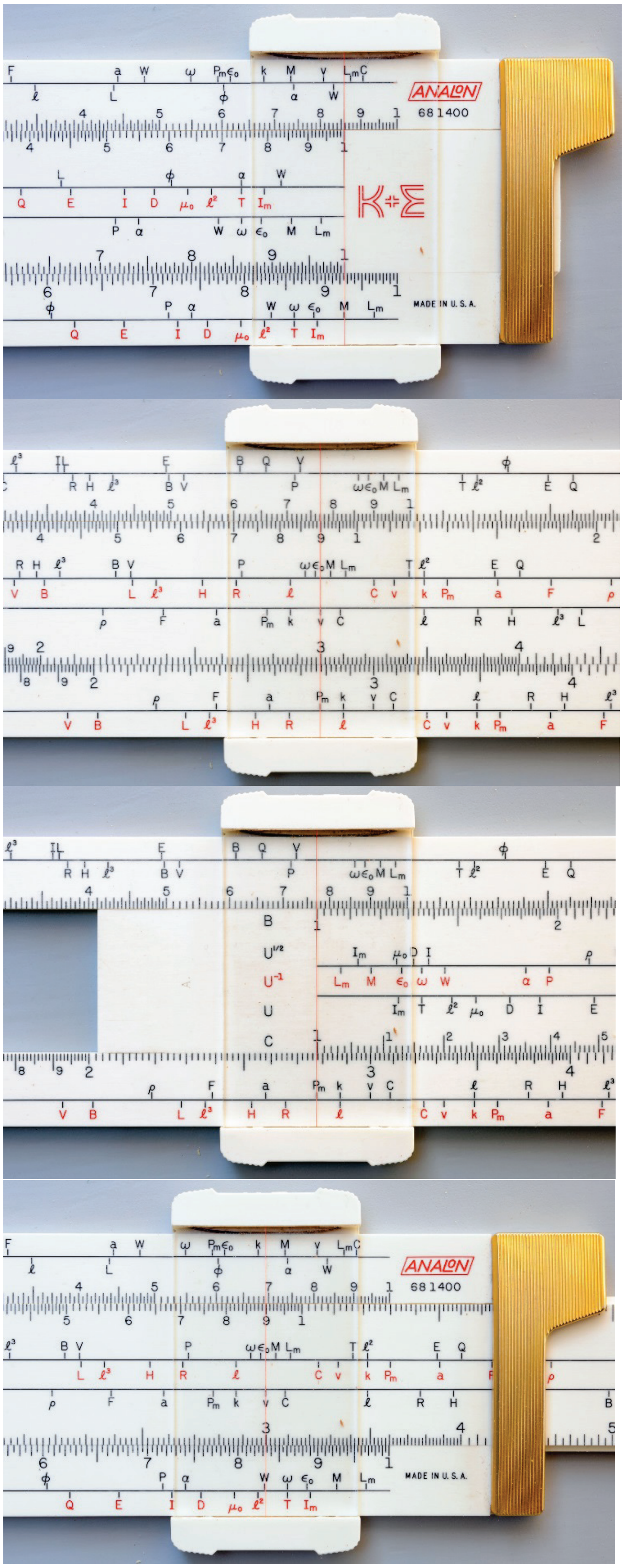

FIGURE 2. Using the Analon To Confirm that mass · velocity · velocity is Energy
(top) Right indicator and hairline are placed at M (mass) on V scale; then (upper middle) hairline moved to v (velocity) on U scale (lower middle). Left indicator is placed at location of hairline on V scale; and (bottom) hairline is again moved to v on U scale; hairline indicates result W on V scale.

On the Analon, the largest blank space is between the values for Q and ρ, separated by distance S_{MAX} = 1.35 units, and situated just to the left of the middle of the rule (see Figure 3). This exceeds an eighth of the total scale length of 10 units, which seems like of an uncomfortably large gap.

An obvious strategy that tends towards accomplishing all three of the above design principles is:

<u>Design Principle #4: For all physical quantities of interest, it is desirable to choose D_L, D_M, D_T, and D_Q so that the ratio S_{MIN}/S_{MAX} is to be as large as possible, or at least "large enough".</u>

How well does the Analon implement Design Principle #4? For the Analon, S_{MIN}/S_{MAX} is ~0.09.

Finally, in K&E's application for a patent for the Analon, Pikus[4] voices an additional design principle:

<u>Design Principle #5: It is desirable that locations on the C and D scales for selected oft-used physical constants correspond directly to the dimensional locations on the U and V scales.</u>

On the Analon, four of the 30 physical parameters on the U and V scales are associated with the locations of fixed constants on the C and D scales (see vertical lines on Figure 3). These are the MKS values for the speed of light (c = $2.998\text{x}10^8$ m/sec), the charge on the electron ($e = 1.602\text{x}10^{-19}$ C [coulomb]), vacuum permittivity[11] (sometimes also called the electric constant; $\varepsilon_o = 8.854\text{x}10^{-12}\ \text{C}^2 \cdot \text{sec}^2/\text{kg} \cdot \text{m}^3$), and the vacuum permeability (sometimes also called the magnetic constant; $\mu_o = 1.257\text{x}10^{-6}$ kg · m/C^2). Moreover, since ε_o, c, and the magnetic permeability μ_o are physically related (i.e., $\varepsilon_o \cdot \mu_o = 1/c^2$), fixing the location for any two of these parameters on U and V scale of a proto-Analon slide rule automatically fixes the location of the third.[12]

Comparing the Analon and Various Hypothetical Dimensional Slide Rules

In order to assess which of the above design principles controlled the Analon design, and also to determine whether other hypothetical dimensional slide rules might have superior designs, I wrote two Fortran programs that constructed numerous hypothetical dimensional slide rules. The first program constructed rules without applying Design Principle #5; the second program applied Design Principle #5. That is, the second program fixed the locations for v, Q, ε_o, and μ_o on the U and V scales at the corresponding locations for e, c, ε_o, and μ_o on the C and D scales.

The first program constructed ten million different hypothetical dimensional slide rules; for each rule the program used a Fortran random number generating routine to assign values between 0 and 10 units to D_L, D_M, D_T, and D_Q. For these ten million hypothetical rules, the program identified the three 'best' rules as specified by Design Principles #2, #3, and #4 (see Table 1 and Figure 3).

TABLE 1. Values of Basis Lengths

For the Analon and several hypothetical slide rules discussed in this article, the values of the basis lengths D_L, D_M, D_T, and D_Q represent the physical parameters of length, mass, time, and electric charge. For the specified basis lengths and the 30 physical parameters labeled on the Analon (see Figure 1 and Table 2), S_{MIN} and S_{MAX} are the minimum and maximum distances separating adjacent labeled parameters. All tabled values are for a rule having a total scale length of 10 units. Tabled values for the Analon are as measured by the author; other tabled values are optimal values for hypothetical dimensional slide rules, calculated as described in the text.

Rule	D_L	D_M	D_T	D_Q	S_{MIN}	S_{MAX}	S_{MIN}/S_{MAX}
A. Analon	5.42	9.66	0.66	2.05	0.12	1.35	0.09
Program 1: D_L, D_M, D_T, and D_Q all vary independently							
B. largest S_{MIN}	6.9207	1.1799	0.1714	5.3675	0.1714	0.8642	0.1983
C. smallest S_{MAX}	3.1865	5.4885	2.7107	4.0992	0.0008	0.4793	0.0016
D. largest S_{MIN}/S_{MAX}	7.7319	9.0642	4.9191	6.6194	0.1339	0.5678	0.2357
Program 2: D_T varies independently; D_L, D_M, and D_Q depend on e, c, ε_o, and D_T							
E. largest S_{MIN}	5.4296	9.6556	0.6613	2.0466	0.1327	1.3202	0.1005
F. smallest S_{MAX}	6.3167	8.7685	1.5483	2.0466	0.0279	0.5867	0.0475
G. largest S_{MIN}/S_{MAX}	6.4276	8.6575	1.6593	2.0466	0.0916	0.8087	0.1133

This program finds numerous hypothetical rule designs that are superior to the Analon with respect to Design Principles #2, #3, or #4, i.e., where S_{MIN} exceeded 0.12 units, S_{MAX} was less than 1.35, or S_{MIN}/S_{MAX} exceeded 0.09. Of the three rule designs that are 'best' by these criteria (see B, C, and D in Table 1 and Figure 3), the design for rule C is not a viable candidate for a working slide rule. Although its S_{MAX} of 0.4793 units was less than half that of Analon's, its S_{MIN} of 0.0008 units is too small to be practical; i.e., the scale locations separating physical parameters v and μ_o, and also L_m and E, nearly coincide. However, the designs for both rules B and D are workable, as both are superior to the Analon with respect to all three Design Principles #2, #3, and #4.

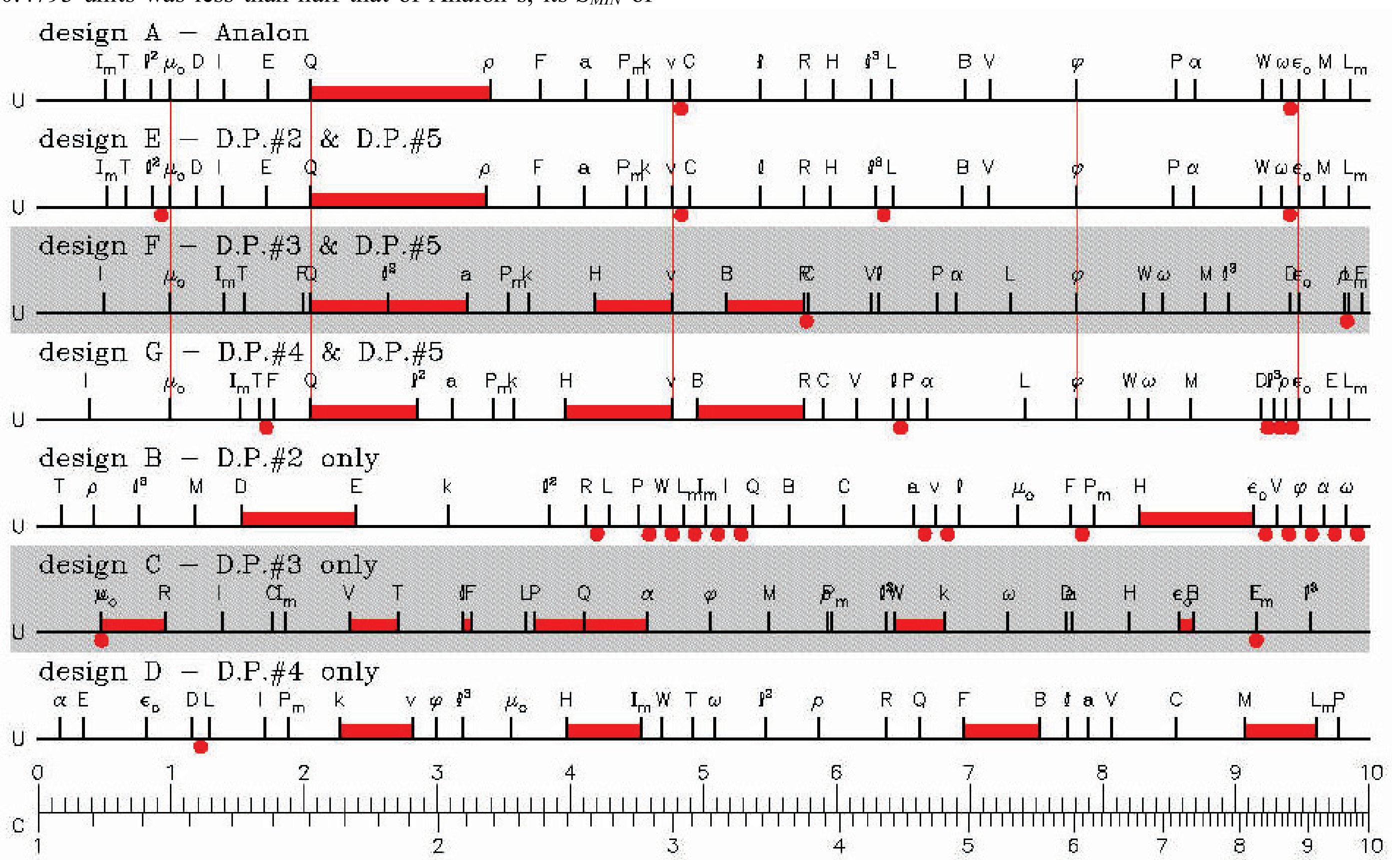

FIGURE 3. Locations of the 30 Labeled Physical Parameters (Table 2) for the Analon and the Six Other Possible Designs

Red shaded areas and red dots indicate the smallest and largest gaps (S_{MIN} and S_{MAX}) between labeled parameters. Designs for the Analon and rules E, F and G fix the locations (red vertical lines) for parameters v, Q, ε_o, and μ_o at values for fundamental physical constants on the C and D scales[12]; rules B, C, and D are designed without fixing any parameters. Note that rule E, designed to maximize S_{MIN}, is virtually identical to the Analon. Note that rules C and F (shaded gray), designed to minimize S_{MAX}, are impractical because smallest gaps S_{MIN} are unreasonably small. However, the rules B, D, and G all represent practical alternate designs for the Analon. Scales at bottom of figure are a conventional logarithmic C scale and a linear rule divided into 10 equal units.

The second computer program constructs hypothetical slide rules by fixing Q, v, ε_o, and μ_o at the same locations as on the Analon rule. The program accomplishes this as follows—from Table 2 and equation (3), the locations for Q, v, and ε_o are:

$$X_Q = D_Q = 10 \log_{10}(1.602) \quad (5)$$

$$X_v = D_L - D_T = 10 \log_{10}(2.998) \quad (6)$$

$$X_\varepsilon = -3D_L - D_M + 2D_T + 2D_Q = 10 \log_{10}(8.854) \quad (7)$$

Solving for D_L and D_M in terms of D_T gives:

$$D_Q = 10 \log_{10}(1.602) \quad (8)$$

$$D_L = D_T + 10 \log_{10}(2.998) \quad (9)$$

$$D_M = -30 \log_{10}(2.998) - D_T + 20 \log_{10}(1.602) - 10 \log_{10}(8.854). \quad (10)$$

Thus, with equation (3) and equations (8) - (10) we can express the locations of all 30 of the physical parameters in Table 2 in terms of the value we choose for D_T.

The second computer program constructs hypothetical rules in this manner for values of D_T at ten million equally-spaced increments between 0 and 10. Of these, the program identifies the three 'best' rules (see E, F, and G in Table 2 and Figure 3) as specified by Design Principles #2, #3, or #4 along with #5. The design for rule F fails as a viable candidate for a working rule; once again the value for S_{MIN} (0.03 units separating physical parameters R and C, and also ρ and L_m) is too small to be practical. However, designs E and G both are viable.

Indeed, design E is virtually indistinguishable from the Analon. This suggests that K&E's design priority was to construct a rule where the labeled parameters were well separated (Design Principle 2: S_{MIN} as large as possible), rather than a rule that 'balanced' separating all labeled parameters, while avoiding large gaps between some of them (Design Principle 4). K&E's design engineers must have decided that rules with S_{MIN} of 0.12 to 0.13 units (rule E and the Analon) were superior to rule G, where S_{MIN} was 0.09-0.11, even though rule G's S_{MAX} was significantly smaller (0.81 units vs 1.35 units).

TABLE 2. Physical Parameters

The 30 physical parameters explicitly marked on the Analon's U and V scales, and their equation (3) coefficients n_L, n_M, n_T, and n_Q with respect to length, mass, time, and charge.

Symbol	n_L	n_M	n_T	n_Q	*Physical Parameter*
a	1	0	-2	0	acceleration
B	0	1	-1	-1	magnetic induction
C	-2	-1	2	2	capacitance
D	-2	0	0	1	electric displacement
E	1	1	-2	-1	electric field
F	1	1	-2	0	force
H	-1	0	-1	1	magnetic intensity
I_m	2	1	0	0	moment of inertia
I	0	0	-1	1	electric current
k	-1	0	0	0	spatial frequency
L	2	1	0	-2	inductance
L_m	2	1	-1	0	angular momentum, action
l	1	0	0	0	length
l^2	2	0	0	0	area
l^3	3	0	0	0	volume
M	0	1	0	0	mass
P	2	1	-3	0	power
P_m	1	1	-1	0	linear momentum
Q	0	0	0	1	charge
R	2	1	-1	-2	resistance, reactance
T	0	0	1	0	time, period
V	2	1	-2	-1	potential (voltage)
v	1	0	-1	0	velocity
W	2	1	-2	0	work, energy, torque
α	0	0	-2	0	angular acceleration
ε_o	-3	-1	2	2	electric permittivity
μ_o	1	1	0	-2	magnetic permeability
ϕ	2	1	-1	-1	magnetic flux
ρ	-3	1	0	0	density
ω	0	0	-1	0	frequency

Discussion and Conclusions

For dimensional side rules, the layout of the scales is not unique. Alternate scale layouts exist that are at least as practical as the U and V scale layouts used on the Analon.

The evidence presented in this paper suggests that the Analon slide rule was designed so that:

(1) Four of the physical parameters (Q, v, ε_o, and μ_o) marked on the U and V scales correspond to values in MKS units on the C and D scales for fundamental physical constants (e, c, ε_o, and μ_o); and

(2) The smallest distance S_{MIN} separating the physical parameters marked on the U and V scales was as large as possible.

The Analon was not designed to make the largest distance S_{MAX} separating the physical parameters marked on the U and V scales as small as possible. Rules based strictly on this principle are impractical because some pairs of physical parameters are too close together, both for rules such as rule F that align selected physical parameters with fundamental constants (as the Analon does), and for rules like rule C that do not so align them.

Also, the Analon was not designed so that the ratio S_{MIN}/S_{MAX} is as large as possible. That is, designs exist with S_{MIN}/S_{MAX} larger than the Analon, both for rules like rule G that align selected physical parameters with fundamental constants (as the Analon does) and for rules like rule D that do not so align them. Both rules G and D appear to be practical as alternate designs for dimensional slide rules.

Was there a high probability that the Analon or similar dimensional slide rules would have become popular if electronic calculators had not been invented? For several reasons I believe the probability was low.

• One reason dimensional analysis is useful is to help discover errors that arise from improperly mixing units (like confusing calories and joules as MKS units of energy, or mixing MKS and cgs units). An analysis with the Analon does not uncover errors of this kind. Thus, in practice one needs to do dimensional analysis on paper anyhow, making the Analon redundant.

• In addition, dimensional analysis on paper is not very difficult, so situations in which the Analon would be very useful are rare.

• Unlike conventional slide rules, for which basic operation relies in a straightforward way on logarithms, the mathematical principles underlying the Analon do not arise naturally from the properties of physical parameters. Understanding how a conventional slide rule works helps one understand logarithms and vice versa; understanding how the Analon works does not help understand relationships between physical parameters.

• An important limitation inherent to dimensional analysis is that, quite often, different physical parameters have precisely the same dimensional units (see also article by Freudiger et al.[13]). For example, energy, torque, and the scalar moment of earthquakes all have the same dimensions (kg · m^2/sec^2 in MKS units), but torque and scalar moment do not correspond physically to energy. A 10-meter racing yacht is unlikely to be 10 meters long, nor to possess any physical dimension that equals 10 meters. There is no obvious way that Analon-style rules serve to mitigate this confusion.

• Using the Analon requires ready access to a table (see Figure 1 and Table 2) that identifies the symbols representing the physical quantities. On the Analon this takes up valuable space on the back of the rule that might have been used for other computational scales; yet if the table were left off the rule the user would have to consult a separate table; either way the additional consulting is inconvenient.

• There is considerable arbitrariness in the selection of the physical parameters marked on the Analon. Why were e, c, ε_o, and μ_o chosen at fixed locations, but not, e.g., the gravitational constant "big G"? Why was temperature excluded as a fundamental parameter? The exclusion means that the Analon is not useful for assessing numerous formulas used by all students of introductory physics and chemistry, such as the ideal gas law ($PV = nRT$).

Nevertheless, the design of the Analon was highly innovative. If calculators had not replaced slide rules, other dimensional slide rules probably would have appeared, possibly targeting specialized audiences, such as chemists and even earthquake seismologists. Undoubtedly slide rule manufacturers would have experimented with other layouts[14] and designs. Because there is a certain 'geek market' attracted to slide rules with numerous scales, dimensional scales-either like or unlike the Analon's U and V scales (Figure 1 and Figure 3) might have been added to some of the more complex slide rules.

However, these events did not happen, and the Analon is unique—there is nothing else like it.

Notes

1. Keuffel & Esser, *Catalog 8: Slide Rules*, published by Keuffel & Esser Company, 1967, page 31.
2. This is perhaps unwarranted, given that there are hundreds of Analons and only one Holy Grail.
3. Soper, Joseph L., *K&E Salisbury Products Division Slide Rules*, published by the Oughtred Society, Pleasanton, CA, 2007, page 131.
4. Pikus, Alfred B., *Calculating device for solving unknown qualitative parameters*, US Patent 3,282,500, 1966.
5. Smyth, Michael P., *K & E Analon Engineering-Science Analysis Slide Rule Instruction Manual*, Keuffel & Esser, New York, NY, 1967, page 32.
6. Smyth, Michael P. and Alfred B. Pikus, *A numerical approach to the relating of physical quantities and dimensional analysis*, IEEE Trans. Education, 11, 1968, pages 63-65.

7. O'Leary, Michael P., *How does the Analon work?*, Journal of the Oughtred Society, 12:1, 2003, pages 18-19.

8. Some subfields of science recognize other fundamental quantities, such as temperature.

9. Actually, my Analon is 24.9 cm or 9.8 inches long, but in this article I use units of 1/10 of this or nominally about one inch, because this arbitrary length simplifies the explanations.

10. In mathematics, the function that ignores multiples of 10 and finds remainders is called 'modulo 10' or 'mod 10' Thus, the proper, accurate notations for equations (3) and (4) are $X = [n_L D_L + n_M D_M + n_T D_T + n_Q D_Q]\ mod\ 10$ and $X_W = [2D_L + D_M - 2D_T]\ mod\ 10$, respectively. In the remainder of this paper I will not write the mod 10 explicitly, assuming that slide rule users are used to finding the answer that lies on the rule and thus will understand how this works.

11. Incidentally, in the table on the back of the Analon, "permittivity" is incorrectly spelled (although still used) as "permitivity".

12. Also, fixing the locations of Q, v, and ε_o on the U and V scales has the consequence of constraining the location for ϕ as well. You can prove this if you use equation (3) and Table 2 to write the expressions for X_Q, X_v, *and* X_ε, and then confirm that $X_\phi = X_Q - X_v - X_\varepsilon$. Obviously, X_ϕ is fixed if X_Q, X_v, and X_ε have fixed values.

13. Freudiger, Shannon, Michael Freudiger, and Robert Koppany, *A qualitative analysis of a quantitative device*, Journal of the Oughtred Society, 9:1, 2000, pages 38-40.

14. On the Analon the U and V scales are separated by the C and D scales. I suggest a better layout would have the U and V scales adjacent, perhaps replacing the A and B scales.