Prosthaphaeresis and Johannes Werner

A history of the forerunner of the logarithm and its inventor[1]

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Introduction

For a long time, people have looked for ways to simplify computing procedures. The difficulty of the calculations was not so important; the goal was always to reduce the amount of computation, but without losing any accuracy.

Particularly in the field of astronomy, in which mathematics first developed, where computations with multidigit numbers were (and still are) a necessity, these computations were very expensive in time and effort. This particularly concerned the basic operations of arithmetic such as multiplication and division, so the possibility to simplify these operations, for example by reducing multiplication to addition, would be an ideal solution.

The most well-known example of this methodology, the reduction of multiplication to simpler functions, is the logarithm; logarithms were published in 1614 in Edinburgh by John Napier (1550-1617) in the first table of logarithms (*Mirifici Logarithmorum Canonis Descriptio*).

But what happened before then? How did astronomers do their calculations without having logarithms at their disposal? The answer is that for about hundred years they used prosthaphaeresis, (also written as prosthaphairesis or prostaphairesis)

Historical perspectives

Literature on the subject of prosthaphaeresis frequently mentions an incorrect name as its inventor; usually the discovery is attributed to the astronomer Tycho Brahe or to his pupil Paul Wittich, or sometimes even to Christopher Clavius. However, Brian Borchers gave a short overview of Prosthaphaeresis and its history in his article in the Journal of the Oughtred Society (JOS) [1], and in that article he referred to its originator as being Johannes Werner.

Borchers' article stands as the starting point for this article, in which the background to the prosthaphaeretic formula, and to prosthaphaeresis itself, will be clarified from historical and mathematical viewpoints.

The term "prosthaphaeresis" - meaning a system in which one uses addition and subtraction - also has other usages in astronomy; thus one speaks, for example, of prosthaphaeresis in connection with: aequinoctiorum; eccentritatis; latitudinis; nodi pro eclipsius; orbis; tychonica; nodorum - "an orbiting body does not move itself evenly; it moves more slowly if the Sun is in the proximity of the body; faster, if the Sun moves away from it. " [2]. However, these purely astronomical usages of the term "prosthaphaeresis" will not be considered further in this essay.

Johannes Werner (1468 - 1522) can be seen to be the discoverer of prosthaphaeresis, and substantial support for this can be found in a work by Axel Anthon Björnbo [3]. As a disciple of the historian of science Anton von Braunmühl, Björnbo took up von Braunmühl's references to some inconsistencies and went to Rome in 1901, so that he could read and study appropriate ancient material in the Vatican library. Of particular interest, he found an undated manuscript, with the title *I. Ioannis Verneri Norimbergensis "de triangulis sphaericis"* in four books, and also *II. Ioannis Verneri Norimbergensis "de meteoroscopiis"* in six books.

Queen Christina of Sweden had been in possession of the manuscript mentioned above, probably between 1654 and 1689; this document had been previously owned by Jakob Christmann (1554 - 1613). After Queen Christina's death in 1689, this manuscript (Codex Reginensis Latinus 1259, i.e. Regina Sveciae Collection, item 1259) lay mainly disregarded in the Vatican. The journey of this manuscript through time is documented in chapter 11 of the main article[1]. During further investigations it became clear that Werner was the editor and/or an author of the two handwritten parts, but that he did not physically write them himself. As to the actual writer of the document, Björnbo identified a mathematically-inexperienced professional scribe of the time [4].

The text of the first complete part of the manuscript (*de triangulis sphaericis*) can be found in Björnbo's work [5] on pages 1 - 133. Later on Björnbo voices his opinions concerning this manuscript both in the "publisher remarks" [Björnbo; Chapter 3] and also in "text history" [6] in a very detailed research report.

An innovation for its time, the organisation of the books concerning spherical triangles is the following [7]:

1. An explanation of the different possible triangle forms (Book I)
   a. A discussion concerning the spherical triangle
2. Solutions of the right-angled triangle (Book II)
   a. The spherical-trigonometric basic formulae
   b. The solution of the right-angled spherical triangle
3. Solutions of the non-right-angled (obtuse) triangle (Book III and IV)
   a. The solution of the obtuse triangle by decomposition into right-angled triangles (Book III)
   b. The solution of the obtuse spherical triangle by a prosthaphaeretically transformed Cosine rule (Book IV)

The three categories stated are of the same structure as the contents of the *Opus Palatinum* of Rheticus of 1596, in so far as spherical triangles are concerned. The extensive contents of the above mentioned manuscript will not be dealt with further in this article.

In chapter 5 [8] is summarised the structure of the contents of the individual books in a tabular form. Thus the findings of Björnbo, namely, the assumptions of Anton von Braunmühl [9] which confirm the authorship of the prosthaphaeretic formula, along with recent up-to-date work by David A. King [10] and Victor E. Thoren [11] together constitute the foundation for the remainder of the main article.[1]

The mathematics behind prosthaphaeresis

To remind the reader, prosthaphaeresis provides a method by the means of which the process of multiplication can be converted into an addition or a subtraction by the use of trigonometric formulae. This technique provided a substantial easing of work for the astronomers of the time.

Looking at books of formulae or on the Internet [12, 13] it becomes clear that there are many ways of expressing the prosthaphaeresis formulae. These formulae, which we know as the "prosthaphaeretic formulae" or as the "prosthaphaeresis formulae", are also known as the "Werner Formulae" or as the "Werner Formulas" (Figure 1).

$$2\sin\alpha\cos\beta = \sin(\alpha-\beta)+\sin(\alpha+\beta) \qquad (1)$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta)+\cos(\alpha+\beta) \qquad (2)$$
$$2\cos\alpha\sin\beta = \sin(\alpha+\beta)-\sin(\alpha-\beta) \qquad (3)$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta) \qquad (4)$$

FIGURE 1.
The Werner Formulae

(In German linguistic usage [14] these formulae shown in Figure 1 are known as "Die prosthaphäretischen Formeln".)

A URL on the website containing Figure 1 leads to the Prosthaphaeresis Formulae as shown below in Figure 2; these formulae are known as "Simpson's Formulae" or "Simpson's Formulas". However these formulae differ in their representation and in their ease of use.

$$\sin\alpha+\sin\beta = 2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right] \qquad (1)$$
$$\sin\alpha-\sin\beta = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right] \qquad (2)$$
$$\cos\alpha+\cos\beta = 2\cos\left[\frac{1}{2}(\alpha+\beta)\right]\cos\left[\frac{1}{2}(\alpha-\beta)\right] \qquad (3)$$
$$\cos\alpha-\cos\beta = -2\sin\left[\frac{1}{2}(\alpha+\beta)\right]\sin\left[\frac{1}{2}(\alpha-\beta)\right] \qquad (4)$$

FIGURE 2.
Prosthaphaeresis Formulae from Mathworld

In more modern collections of formulae these names are not used, but instead the formulae are referred to as "products of trigonometric functions" [16] - see Figure 3.

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$
$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta)+\cos(\alpha+\beta)\right]$$
$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta)+\sin(\alpha-\beta)\right]$$
$$\cos\alpha\sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta)-\sin(\alpha-\beta)\right]$$

FIGURE 3.
The Prosthaphaeretic Formulae as "products of trigonometric functions"

As a first application, an example of a multiplication of the numbers 0.6157 and 0.9397 is shown with one of the formulae shown in Figure 3 above, here restated:

$$A \bullet B = \sin a \bullet \cos b = \frac{1}{2}\left[\sin(a+b)+\sin(a-b)\right]$$

with the factors A = sin *a* = 0.6157 and B = cos *b* = 0.9397 [15]. From the table in Fig 4 we read for factor sin *a* an angle of *a* = 38° in the green grads (degrees) column, and for cos *b* an angle of b = 20° from the red inverted grads (degrees) column (see red ellipses).

Grad	sin	tan	
0	0,0000	0,0000	90
1	0175	0175	89
2	0349	0349	88
3	0523	0524	87
4	0698	0699	86
5	0872	0875	85
6	1045	1051	84
7	1219	1228	83
8	1392	1405	82
9	1564	1584	81
10	0,1736	0,1763	80
11	1908	1944	79
12	2079	2126	78
13	2250	2309	77
14	2419	2493	76
15	2588	2679	75
16	2756	2867	74
17	2924	3057	73
18	3090	3249	72
19	3256	3443	71
20	0,3420	0,3640	70
21	3584	3839	69
22	3746	4040	68
23	3907	4245	67
24	4067	4452	66
25	4226	4663	65
26	4384	4877	64
27	4540	5095	63
28	4695	5317	62
29	4848	5543	61
30	0,5000	0,5774	60
31	5150	6009	59
32	5299	6249	58
33	5446	6494	57
34	5592	6745	56
35	5736	7002	55
36	5878	7265	54
37	6018	7536	53
38	6157	7813	52
39	6293	8098	51
40	0,6428	0,8391	50
41	6561	8693	49
42	6691	9004	48
43	6820	9325	47
44	6947	9657	46
45	7071	1,0000	45
	cos	cot	Grad

Grad	sin	tan	
45	0,7071	1,000	45
46	7193	036	44
47	7314	072	43
48	7431	111	42
49	7547	150	41
50	0,7660	1,192	40
51	7771	235	39
52	7880	280	38
53	7986	327	37
54	8090	376	36
55	8192	428	35
56	8290	483	34
57	8387	540	33
58	8480	600	32
59	8572	664	31
60	0,8660	1,732	30
61	8746	804	29
62	8829	881	28
63	8910	963	27
64	8988	2,050	26
65	9063	145	25
66	9135	246	24
67	9205	356	23
68	9272	475	22
69	9336	605	21
70	0,9397	2,747	20
71	9455	904	19
72	9511	3,078	18
73	9563	271	17
74	9613	487	16
75	9659	732	15
76	9703	4,011	14
77	9744	331	13
78	9781	705	12
79	9816	5,145	11
80	0,9848	5,671	10
81	9877	6,314	9
82	9903	7,115	8
83	9925	8,144	7
84	9945	9,514	6
85	9962	11,43	5
86	9976	14,30	4
87	9986	19,08	3
88	9994	28,64	2
89	9998	57,29	1
90	1,0000	∞	0
	cos	cot	Grad

FIGURE 4.
Four-figure table from [17]

That is, for the two figures in the blue boxes we see the results for $a + b = 58°$ and for $a - b = 18°$. With sin 58° = 0.8480 and sin 18° = 0.3090 their sum is 1.1570. Halving this gives 0.5786, which is the correct product of 0.6157 • 0.9397 to 4 places (to be more exact: 0.57857329).

It is easy to recognise that using a sine table with higher precision will result in a higher accuracy. Using Regiomontanus' seven-figure tables from his sine tables of 1541 we find sin 18° = 0.3090170 and sin 58° = 0.8480481, giving a sum of 1.1570651; halving that value gives a result of 0.578532550 – a value now quite close to the real product.

An introduction to Johannes Werner

Johannes Werner was born on 14 February 1468 in Nuremberg and died in (May?) 1522 in Nuremberg while he was in the position of parish priest in the municipality of St. Johannes.

Werner studied theology and mathematics in Ingolstadt from 1484 onwards. In 1490 he became a chaplain in Herzogenaurach.

From 1493 to 1497 he lived and worked in Rome.

In 1503 he was appointed as the vicar at the church in Wöhrd, a suburb of Nuremberg.

Kaiser Maximilian I appointed him the Imperial Chaplain.

Later, he became a priest at the Johanniskirche in Nuremberg; he held this position up to his death.

Astronomers have honoured him by naming a crater on the moon, "Werner", after him.

FIGURE 5.
Johannes Werner

Endnote

1. The **JOS PLUS** logo indicates that further information relating to this article can be found on our website http://www.oughtred.org. In this case, a much longer article, going into much more detail, can be found. This longer article is an update and translation of a German-language original which can be found at [18].

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7. Björnbo; Chapter 1 and pages 163; 177-183.
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