Square Root of 10 ($\sqrt{10}$) Folded Scales

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Recent discussions on the International Slide Rule Group (ISRG) web site regarding $\sqrt{10}$ -folded scales prompted me to investigate further the unique properties of these rules, even though they were rather limited in their use, particularly at the end of the slide rule era.

Folded scales, CF, DF and CIF, were introduced by K&E in about 1900 [1]. These were used to reduce the amount of movement of the cursor and slide for computations. At first, these scales were folded at $\sqrt{10}$ but were later folded at π so that π could be used as a factor without resetting or using a gauge mark. It is not clear that K&E understood the unique properties of $\sqrt{10}$ folding, given K&E's description of them. Most slide rule makers used π folding until the end of the slide rule era. A few makers used $\sqrt{10}$-folded scales – the Unique Dualistic and Flying Fish rules are good examples.

Scales can be folded at different values depending on the application. Business rules may have scales folded at 360 to aid in interest computations for bonds or loans and other applications. Pickett introduced rules with scales folded at ln(10) or about 2.303 to aid in natural log computations; these are labeled CF/M and DF/M.

Most user manuals and books show how to use folded scales in order to reduce movements by taking advantage one of the properties of folded scales: for any setting of the slide, a number '*n*' of the C scale opposite a number '*m*' on the D scale then the number '*n*' of the CF scale is opposite the number '*m*' on the DF scale. If a C index cannot be reached by the hairline, most often it can reached on the CF scale at CF index in the center of the rule. If a computation calls for a number that is off-scale on the C scale, it can be found on the CF scale index. This allows for a continuous sequence of operations without moving the slide and re-entering numbers. This technique can be used on any slide rule with folded scales regardless of the folding factor. Most experienced users are familiar with this technique and beginners should learn it as it saves a significant amount of time.

Keep in mind that this technique preserves the equivalent ratios between the folded scales and the fundamental scales and does not call for the 'projection' of a setting from one group to the other. The folded scales may be used instead of the fundamental scales and vice versa. That is, the user may move from one group of scales to another within a computation or between computations. The user may or may not be advised not to mix the scales in one group with scales in the opposite group as this will result in an error if the folding factor is other than $\sqrt{10}$. K&E describes the constraint [1] as:

> [In combined operations], the scale of operation may be changed at will from the C scale to the CF scale or vice versa. In general, however, if the answer is read on the D scale, the number of times the hairline has been pushed to a mark on CF[->CF()] must be the same as the number of times a mark on CF[=>CF()] has been drawn under the hairline. If the answer is read on DF, the process of pushing the hairline to a number on DF must have been used exactly one more time than the process of drawing a mark on CF under the hairline.

Another property of folded scales is this: any setting of a number on the D scale results in the product of the folding factor and this number on the DF scale and the setting of any number on the DF scale results in 1/(folding-factor) on the D scale. I will call this process 'projection' after Snodgrass [3]. For π-folded rules, setting the hairline to D(5) yields DF(157) under the hairline while the projection of DF(5) yields D(158). For $\sqrt{10}$-folded scales, D(5) yields DF(168) and DF(5) yields D(158).

To see how the unique $\sqrt{10}$-fold factor works, consider the projection of x from the D scale to the DF scale and the projection of x from the DF scale to the D scale.

$$D(x) >> DF\left(x \bullet \sqrt{10}\right)$$

$$DF(x) >> D\left(\frac{x}{\sqrt{10}}\right) >> D\left(\left(\frac{x}{\sqrt{10}}\right) \bullet \left(\frac{\sqrt{10}}{\sqrt{10}}\right)\right) >> D\left(\frac{\left(x \bullet \sqrt{10}\right)}{10}\right)$$

The x on the scale is the mantissa of the number and the division by 10 can be considered to be absorbed by the 'characteristic' or decimal point of the number x. This yields

$$D(x) >> DF\left(x \bullet \sqrt{10}\right)$$
$$DF(x) >> D\left(x \bullet \sqrt{10}\right)$$

This would allow the interchange of D(x) for DF(x) as well as C for CF, and CIF for CI or vice- versa without the C-scale constraint described above for conventional folded scale use.

From time to time, one sees comments that a slide rule with $\sqrt{10}$-folded scales has a slight advantage in doing computations but no details are provided. This advantage comes from the fact that one scale setting can be replaced by using the opposite group scale setting without changing the result. An example will show how this works.

This example from the FF 1002 user's manual calls for finding the quotient of 248/7.25. The solution mixes scales from both the fundamental scales and the folded scales.

-> D(248)	set hairline to 248 on D scale
=>CF(725)	move slide to 725 on CF scale under hair line
->CF($\sqrt{10}$)	set hairline to left CF index
*DF(342)	read result under hairline on DF scale.

Continuing, you can get the same result by these methods:

->C(1)	to read the result on the DF scale.
*DF(342)	read result on the DF scale
->CF(10)	to read the result on the D scale
*D(342)	read the result on the D scale

All of this is possible due to the D to DF projections as derived above.

This sequence of operations will not work on a π-folded slide rule. What has been done here is to cross from the C scale in the second step to the CF scale to continue the operation. (Note that C(725) lines up with DF(248) and the correct result lines up with CF(1) on the D scale as described above for conventional use.) Mechanically, the third step will yield the correct result but should read ->C(1) for a more technically correct setting. Table 1 shows how the π-folded rule gives incorrect answers using the same settings as above.

TABLE 1.
π-Folded Rule Errors

π-fold	$\sqrt{10}$-fold
->D(248)	->D(248)
⇒CF(75)	⇒CF(75)
->C(1)	->C(1)
*DF(338) wrong	*DF(342) correct
->D(248)	->D(248)
⇒CF(75)	⇒CF(75)
->CIF(π)	->C(1)=CF($\sqrt{10}$)
*DF(342) correct	*DF(342) correct
->D(248)	->D(248)
⇒CF(75)	⇒CF(75)
->CF(1)	->C(1)
*DF(342) correct	*DF(342) correct

To use the folded scales in an effective manner called for developing methods that took advantage of them in a consistent way beyond the substitution method as described in most manuals as described above.

Toward the end of the slide rule era, two writers introduced alternate methods for using the folded scales. Johnson introduced his "center-drift" method of high-speed computation in 1949 in the book The Slide Rule [2]. While Johnson doesn't describe it this way, the purpose of the center-drift method is to reduce the total movement or distance the hairline or cursor have to move in doing multiplication and division thus speeding computation. Johnson's method was explicitly designed for π-folded slide rules as they were the most commonly available type at the time.

Snodgrass introduced a $\sqrt{10}$ slide rule in about 1955 with the Unique Dualistic High Speed slide rule. He described the use of this rule in the associated manual but also in more detail in his book Slide Rule in 1955 [3]. His purpose also was to show shorter methods of doing multiplication and division. Clason, in 1964, described Snodgrass' method in a bit more detail in his book *Delights of the Slide Rule* [4]. Both suggest that the $\sqrt{10}$-folded scales are superior to π-folded scales but fail to present any evidence except for the symmetry of the scale projection described above.

The major difference in the two methods is the selection of the set of scales to be used for the next factor in the computation. Johnson chooses the scale depending on the first digit of the factor. If the factor begins with either a 2, 3, 4, 5, use the fundamental scales C, D, or CIF other wise, use the folded scales. Snodgrass says to use the scale where the factor is closer. Both show (and this is common to all slide rules) that all operations start on a D scale and end on the same D scale.

Snodgrass [3] attempts to show the advantages of the $\sqrt{10}$-folding factor in a table explaining slide rule setting using mixed scales in multiplying 3 times 8. He describes six combinations of the scales C, D, CF, DF (there are actually more if CI scales are used). I have modified his table to show the slide rule settings so that these are shown in columns in order left to right. The columns are labeled by the slide rule actions: a single line arrow says to move the hairline to the scale setting in each row. The double line arrow says move the slide to the scale setting shown in each row. The asterisk column says read the result shown in each row.

TABLE 2.
Mixing Scale Groups Example

→	→	⇒	→	*	NOTE
1	DF(8)	CF(1)	CF(3)	DF(24)	1
2	DF(8)	CF(1)	C(3)	D(24)	1
3	D(8)	C(10)	C(3)	D(24)	1
4	D(8)	C(10)	CF(3)	DF(24)	1
5	D(8)	CF(1)	C(3)	DF(24)	2, 3
6	DF(8)	C(10)	CF(3)	D(24)	2

Notes:

1. These group selections are typical folded-scale uses. They do not mix groups and will work regardless of folding factor as you can see with your π-folded slide rule.

2. These settings will work correctly for $\sqrt{10}$-folded scales and will not work for π-folded scales.

3. Row 5 will work with π-folded scales using the Johnson's center drift method. Modify the order as shown here:

->D(3)
=>CF(1)
→CF(8)
*D(24)

Note that this set of settings has shorter cursor and slide motion than either set 5 or 6 using Snodgrass' settings.

Conventional multiplication and division uses the fundamental scales CI, C and D. The computation can be moved to the folded scales CIF, CF and DF to continue the computation of values that fall off scale. Johnson describes this in detail on [2]. He then describes how the two groups of scales can be combined but recognizing that certain combinations cannot be used. These constraints, he acknowledges without explanation, applies to π-folded scales only. In practice, they apply to any folded scale slide rule.

To show the error that results when scale groups are mixed on a □-folded rule, multiply 3×4 by this sequence:

->D(3)
=>CF(1)
->C(4)
*DF(11.84)

Figure 1 shows the physical results on the rule scales. Figure 2 shows the correct result on a $\sqrt{10}$-folded rule.

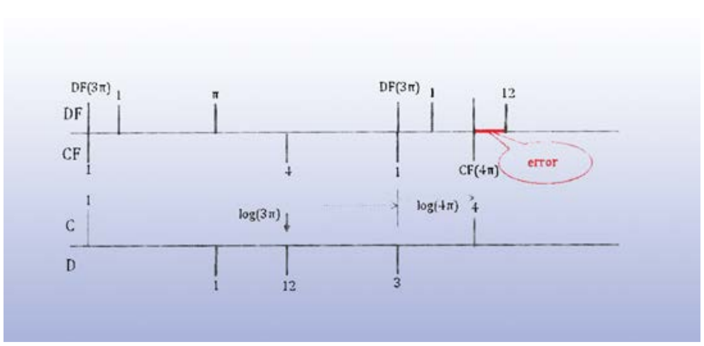

FIGURE 1.
Slide rule settings for □-folded rule showing mixed scales for 3×4 =12.

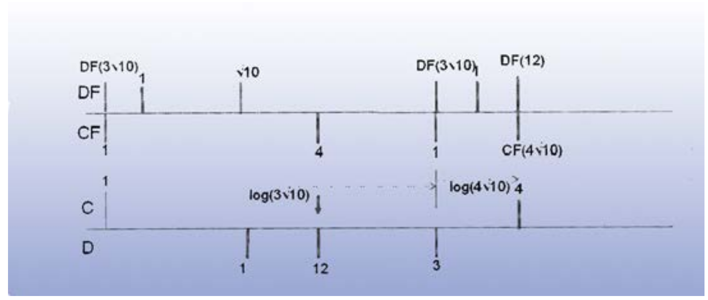

FIGURE 2.
Slide rule settings for $\sqrt{10}$-folded rule showing mixed scales for 3×4 =12.

An example of how mixed scales can be used for mixed operations is given in Johnson page 79 – multiply 2.5 by 1.4. The slide rule operations are:

->D(250)
=>CIF(140)
->C(1)
*D(3.5)

Notice that the fold factor is irrelevant in this example as you can test by using a $\sqrt{10}$-slide rule. What is different is that the two C-factors are chosen from the same scale – either C if it is used first or CF if it is chosen first (the constraint described by K&E) and the result is read from the first chosen D or DF scale. Snodgrass for some reason states that this as a limitation perhaps because it may choose a factor that is not the closer to the hairline.

Classic use of the folded scales does not project values. The folded scales serve as substitutes for the fundamental scales. A ratio on the fundamental scales is duplicated on the folded scales and vice versa. As an example, multiply 4.5x2.9.

→D(45)	→D(45)
=>CI(29)	=>CI(29)
→C(1)	→CF(1)
*D(131)	*DF(131)

Both versions are correct but the right-hand version has shorter movement as the initial settings are in the middle of the rule as is the CF index.

In order to compare of the performance of π-folded scales with $\sqrt{10}$-folded scales, one has to do detailed physical measurements. This can be a bit tedious, as I will show below. Use this common expression from ISRG message 38188 as a basis for comparison:

$$\frac{8.16\times10.4}{2.85\times3.29}=9.05$$

The specific steps vary due to the nature of each solution method. The $\sqrt{10}$-folded solution is taken from that message which appears to be based on Snodgrass' method while I used a center-drift method for the □-folded solution.

TABLE 3.
Calculation of Total Hairline and Slide Movement for Pickett 1010-SL

STEP	->start	->stop	⇒ start	⇒ stop	Δ
->DF(104)	.502	.520			.018
⇒CIF(816)			.502	.432	.070
->CF(1)	.432	.432			.000
⇒C(329)			.432	.420	.012
->CI(285)	.420	.460			.040
*DF( )					Σ=.140

The L scale is on the body of this rule. Use the L scale to measure the distance the hairline moves by recording the L-scale setting under the hairline at the start and stop of the hairline move. To measure the slide movement, because the slide moves with respect to the L scale, record the start and stop setting by moving the hairline to the center index C(1), read the start setting on the L scale, move the slide, repeat the reading under the hairline, and then restore the hairline to the prior setting for the next step.

TABLE 4.
Calculation of Total Hairline and Slide Movement for Flying Fish 1003

STEP	->start	->stop	⇒ start	⇒ stop	Δ
->DF(816)	.500	.412			.088
⇒C(285)			.500	.454	.046
->CF(104)	.454	.518			.064
⇒C(329)			.542	.542	.000
->CIF(1)	.542	.500			.042
*DF( )					□=.240

The L scale is on the slide on this rule so the slide movement measurement is a bit messier. Use the L scale to measure the distance the hairline moves by recording the L-scale setting under the hairline at the start and stop of the hairline move. To measure the slide movement, because the L scale is fixed with respect to the slide, record the start and stop setting by moving the hairline to the center index D(1), read the start setting on the L scale, restore the hairline, move the slide and repeat the measurement for the stop setting. Then, restore the hairline to the prior setting for the next step.

The $\sqrt{10}$ scale solution has a total movement of 60 mm while the center-drift method has a total movement of 36 mm. The difference in the two solutions is not substantial but shows that the $\sqrt{10}$-folded is not always better as might be claimed by some.

I won't show the conventional folded-scale use results but you will see that it is quite a bit longer in cursor and slide movement than either Johnson's or Snodgrass's method. One solution is:

->DF(816)
⇒CF(829)
->C(104)
=C(285)
->CF(1)
*DF(905)

Small examples do not show the wide variation that can occur in real applications. Clason, presumably, in an attempt to show the efficiency of $\sqrt{10}$-folded scales, presents a 10-factor example of mixed multiplication and division:

$$\frac{826\times 0.045\times 4.66\times 60.5\times 0.588}{7.36\times 1.94\times 5.4\times 0.095\times 8.75}$$

He recommends using a work sheet to keep track of the decimal count for determining the decimal point location in the result and using an x format to find the location result. Where Clason uses some 50 or so lines to describe the solution, I will use my shorthand notation to show the individual steps so that I can compare them to Johnson's center-drift solution. I did not try to do any optimization as Johnson suggests but used the factors in sequence calculating the numerator first followed by the denominator. Clason alternated division followed by multiplication — the most conventional method for such problems.

TABLE 5.
Comparison of Both Methods for 10-Factor Computation

SNODGRASS SOLUTION	JOHNSON SOLUTION
→DF(826)	→DF(826)
⇒C(736)	⇒CI(45)
→C(45)	→C(466)
⇒C(194)	⇒CF(605)
→C(466)	→C(588)
⇒C(54)	⇒CF(736)
→C(605)	→CIF(194)
⇒C(95)	⇒C(54)
→C(588)	→CIF(95)
⇒C(875)	⇒CF(875)
→C(10)	→CF(1)
*DF(96+)	*DF(96+)

Total cursor and slide motion for Clason's solution is about 717 mm or 28.2'' while for Johnson's solution is about 602 mm or 23.7''.

After studying and comparing the performance of both techniques (using a Flying Fish 1003 and Pickett 1010-SL) on many examples from [3] and [4], I come to the conclusion that the $\sqrt{10}$-folded scales offer little or no advantage over Johnson's center-drift method. It is possible that Snodgrass was comparing the $\sqrt{10}$-folded scales performance with the conventional folded scale slide rule as in [1]. He may not have been familiar with Johnson's methods even though his book was in print for about 20 years. (My copy found on the used book market came from Bangkok so it must have had world-wide distribution.) It is possible, depending on the specific factors, that Snodgrass' scale selection method, choosing the closest factor, suffers from excessive cursor and slide excursions beyond the central part of the scales. Johnson's scale selection method tends to keep the factors centrally located. In addition, Snodgrass and Clason seem not to use the inverted scales CI and CIF as much as they should have as these tend to reduce cursor and slide motion and steps in the solution.

Though not directly related to the motion issues, Snodgrass' method of doing digit and number of folded-fundamental-scale-group-exchanges accounting by pencil costs the user much more time than would be saved by the scale selection optimization. Johnson assumes the user knows the starting D scale and the result decimal location can be found by classic approximation, thus saving time.

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Marion Moon is a retired engineer. He has, over the past 15 years, been collecting "flagship" slide rules from various makers from around the world. He has been studying the makers' recommendations of the various scale arrangements used. This article is the result of one of those studies.