Diagonals and Transversals: Magnifying the Scale

Otto van Poelje

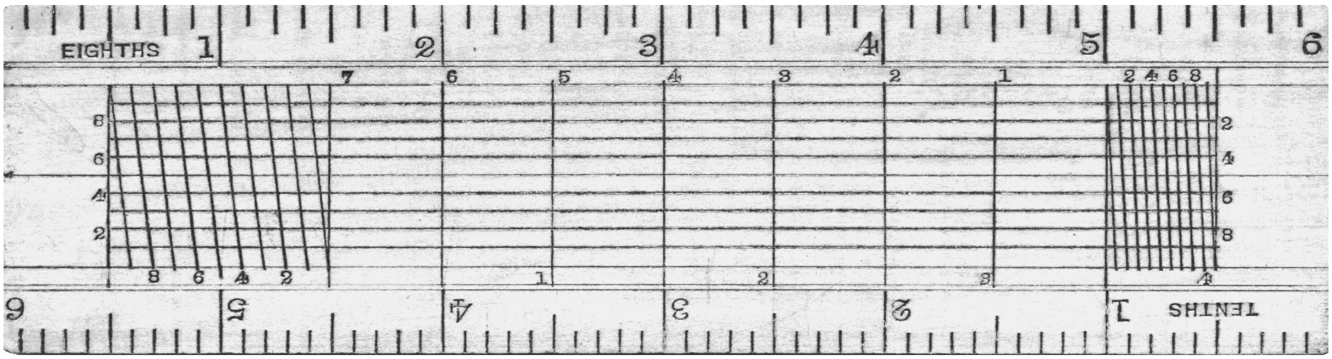


Figure 1. Double diagonal scale on a Reeves & Sons protractor No. 37.

Introduction

During my studies of the Gunter rule, I always passed quickly over the diagonal scales on the front ("transversaalschaal" in Dutch), expecting no surprises in such an elementary drawing tool until my attention was drawn to some specimens with apparent drawing errors in that area. Gunter rules have non-linear scales for goniometrical constructions and calculations, but also linear scales: these are called scales of "Equal Parts" (E.P.), and in most cases they can be recognized by a "forerunner" of a finer division than the full scale itself, see Figure 2.

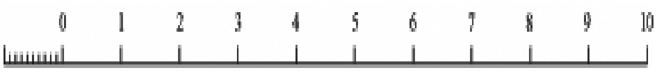


Figure 2. Equal Parts scale with forerunner containing the finer division.

One of the underlying reasons (for short-length finer division) was that production costs of manually engraved scales would be much higher if the finer division was applied over the full length of the scale.

In usage, a measure of, for example, 1.2 extends between two fine units to the left of the zero mark (the forerunner) and one unit to the right of the zero mark (the main scale).

Accuracy and precision of scale markings, on any type of rule, has always been a matter of concern for both maker and user. In this paper we shall not address the manufacturing aspects of the rules for increasing accuracy, but discuss mainly the design of scales to improve the scale-reading precision beyond the physical limits to fineness and visibility of scale markings. These limits are defined by the width of a tick mark that can be engraved, but also by the average human eye, which has problems reading marks thinner than 0.05 mm or when the density of the marks is too high.[1]

Magnifying the scale

One favorite method to improve scale precision can be summarized as "magnifying the scale". Taken literally, a very fine scale division can be read with a magnifying glass, like those fitted to some type of slide rules or nautical sextants. Microscopes can even be equipped with eyepiece micrometers (reticules) or stage micrometers (glass ruler on object plane) for extreme precision. In case of measurement devices with an index pointer, the scale can be magnified by "leverage" of the pointer: for example, Tycho Brahe's great mural quadrant at his Uraniborg observatory had a circular scale of some 7 feet radius for that purpose (1582). In this paper, however, we will focus on magnifying the scale by adding enlarged scales, either in a lateral direction (diagonal, transversal) or in the same direction (Vernier).

Usage of diagonal scales

Diagonal scales occur not only on Gunter rules, but also on many other rules or drawing instruments, like carpenter rules, plain scales, plotting scales and protractors. These scales are used with a set of dividers to either construct or to measure distances in the span between the tips of the dividers. The dividers are used to move a distance between diagonal scale and the real world, like a drawing, map or any physical object. In its basic form the diagonal scale allowed a distance to be measured in three significant digits.

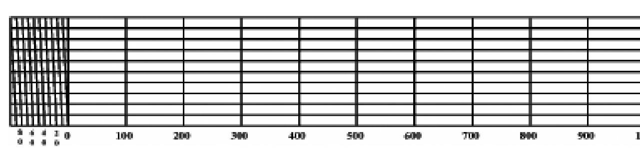


Figure 3. Basic diagonal scale

Figure 3 shows a scale of 1000 units, with hundreds to the right of point zero and tens to the left. Each of the tens should in principle be divided again in 10 units (most significant digits), but space does not really allow this. Therefore the units within the tens are projected vertically upwards onto the respective horizontals, which

[1]For example, try reading a scale containing hundredth-inch marks with the naked eye!

are numbered 1 to 10. A diagonal is then drawn through all intersections of the vertical projection of a unit and the horizontal line bearing that unit's number.

This diagonal can now be used to determine the unit value of a distance: for example, unit 5 (of 25) in Figure 4 projects along the diagonal onto horizontal No. 5, which is much easier to read due to the "magnification" by the slope of the diagonal (actually the horizontal divisions are multiplied in the vertical direction by the tangent of the slope).

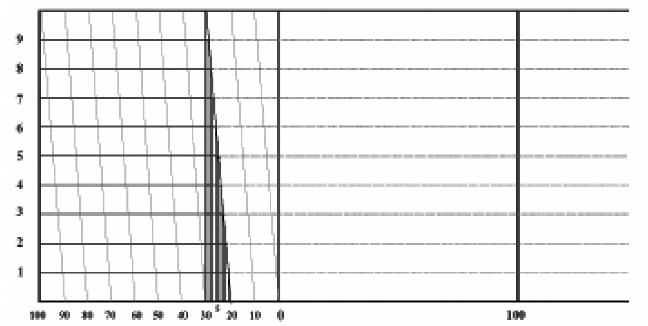


Figure 4. Diagonal as vertical projection of units.

As an example, Figure 5 shows the dividers hovering above a demonstration distance of 125 units, composed of the summation of one hundreds unit, two tens units, and five units as projected on the diagonal between 20 and 30 (see the distance between the two small circles).

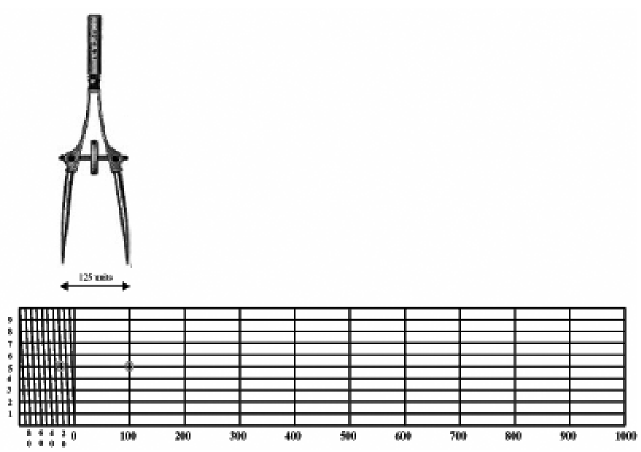


Figure 5. Diagonal scale in operation.

For construction, the right tip of the dividers is put on 100 of horizontal 5 (right circle), the left tip is put on the intersection of the same horizontal and the diagonal between 20 and 30 (left circle).

For measuring this distance of 125 units, the tips should be placed such that the left is in the diagonal area while the right tip is on a hundreds vertical. Then the dividers as a whole should be moved up or down tentatively along the vertical until reaching a horizontal, where the left tip touches the intersection of that horizontal and a diagonal (left circle), while the right tip is on a hundreds vertical (right circle).

Some mixed observations

- This use of the dividers on the diagonal scale can be considered to be a calculation of the formula

$$1 * 100 + 2 * 10 + 5 * 1$$

In reference [15] this principle is extended, for educational purposes, to general additions and subtractions on the diagonal scale.

- This example addresses a scale with three significant digits. Actual scales may have units in length measures (cm, inches, etc), or in scale factors (1:1000, 1:1250, 1:2000 etc).
- This example addresses decimal divisions, but any other division could be used, for example "eighth", "twelfth" (see Figure 6) or "sixteenth" divisions in case of pre-metric measures. Only the amount of horizontals, and their numbering, needed to be adapted accordingly.
- Diagonals are not only used with dividers on straight rules, but can also occur on scales which are read via an index pointer (measurement devices like quadrant or octant, see Figure 10) or a cursor (on slide rules, see Figure 9).
- The positioning of the diagonals in the "forerunner box" is an elegant solution for usage with dividers, minimizing the number of diagonals engraved. Still, there have also been scales with diagonals or transversals on *all* main units. In the case of reading by index pointer this was even essential, see Figures 10 and 13.

V-type diagonal scale (transversals)

Another form of diagonal scale handles only two significant digits, see Figure 6. These diagonals look like an upside-down "V", and are also called "transversals". The advantage is that the height of the scale can be smaller, every horizontal being used twice. In many cases (like Figure 6) the slope of the transversals is so low that there is no significant improvement in precision.

Composite diagonal scale

When rules needed multiple diagonal scales, for different scaling factors, it was usual practice to combine two of them for reasons of space efficiency. Gunter rules have two diagonal sets combined in one scale. Architects' or surveyors' diagonal scales have four differently scaled diagonal sets, two combined on each side.

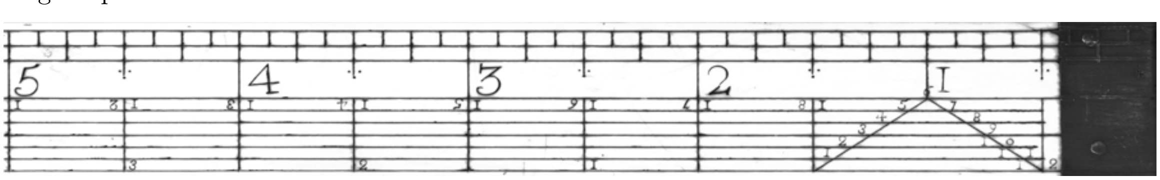


Figure 6. V-type diagonal scale on an ivory carpenters' rule (transversals).

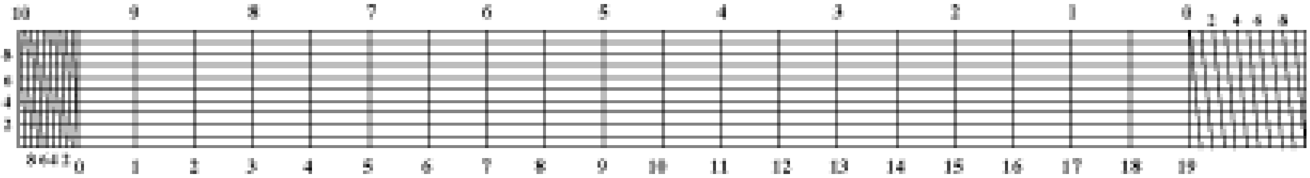
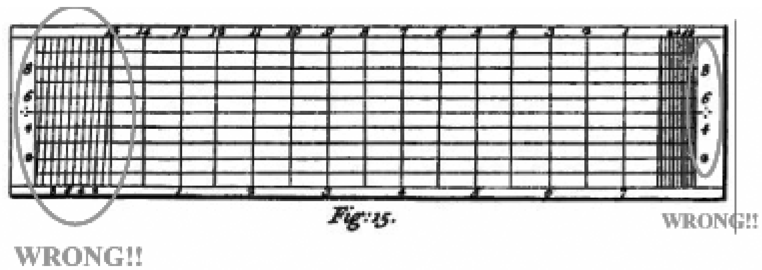


Figure 7. Correct drawing of double diagonal scale.



See Figure 7 for an example of a combined scale on a Gunter rule: one for inches (top scale and right diagonals box), and one for half-inches (bottom scale and left diagonals box). Because the one-inch diagonals are at the right side, the inches themselves are numbered from right to left.

Note that, because of the methods described in the usage section, the diagonals have to slant out from point zero, resulting effectively in diagonals that have the same slanting direction in the left and the right box.

Diagonal scale descriptions

Looking for documentation on diagonal scales, we can find many descriptions in 20^{th} century handbooks on technical drawing, architecture, surveying or machine tooling, for example a "user manual" for a single diagonal scale in the 1947 *Fowler's . . . Pocketbook* [9], page 45-46 (this booklet, by the way, also contains a manual and many advertisements for the well-known Fowler's disc calculator). It is remarkable that for older drawing instruments neither Bion [5] nor Hambly [12] give a satisfactory description of the usage and the resulting design of the double diagonal scale.

One of the best textual descriptions from the past is in Robertson's 1775 *A Treatise of Mathematical Instruments* [6], see the following excerpt:

> **Of the Lines of Equal Parts**
>
> *LINES of equal parts are of two sorts, viz. simply divided, and diagonally divided. Pl. V.*
>
> *Simply divided* . . . .
> (describes the single scale of a common rule)
>
> *Diagonally divided. Draw eleven lines parallel to each other, and at equal distances; divide the upper of these lines into such a number of equal parts as the scale to be expressed is intended to contain; and from each of these divisions draw perpendiculars through the eleven parallels, (Figure 15) subdivide the first of these divisions into 10 equal parts, both in the upper and lower lines; then each of these subdivisions may also be subdivided into 10 equal parts, by drawing diagonal lines; viz. from the 10th below, to the 9th above; from the 9th below to the 8th above; from the 8th below to the 7th above, &c. till from the 1st below to the 0th above, so that by these means one of the primary divisions on the scale, will be divided into 100 equal parts.*
>
> *There are generally two diagonal scales laid on the same plane or face of the ruler, one being commonly half the other. (Figure 15.)*
>
> *The use of the diagonal scale is much the same with the simple scale; all the difference is that a plan may be laid down more accurately by it; because of this, a line may be taken of three denominations; whereas from the former, only two could be taken. Now from this construction it is plain, if each of the primary divisions represents 1, each of the first subdivisions will express 1/10 of 1; and each of the second subdivisions, (which are taken on the diagonal lines, counting from the top downwards) will express 1/10 of the former subdivisions, or a 100th of the primary divisions; and if each of the primary divisions express 10, the each of the first subdivisions will express 1, and each of the 2d, 1/10; and if each of the primary divisions represent 100, then each of the first subdivisions will be 10; and each of the 2d will be 1, &c. Therefore to lay down a line, whose length is express'd by 347, 34 7/10 or 3 47/100 whether leagues, miles, chain, &c. On the diagonal line, joined to the 4th of the first subdivisions, count 7 downwards, reckoning the distance of each parallel 1; there set one point of the compasses, and extend the other, till it falls on the intersection of the third primary division with the same parallel in which the other foot rests, and the compasses will then be opened to express a line of 347, 34 7/10, or 3 47/100, &c.*

The most concise description, however, is to be found in Heather's book [7]:

> *"General Rule to take off any Number of Three places of Figures upon a Diagonal Scale: On the parallel indicated by the third Figure, measure from the diagonal indicated by the*

second Figure to the vertical indicated by the first."

Errors in drawing the diagonals

When we look at Robertson's "Figure 15", the top scale presents half-inches, right to left, with the correct diagonals set at the right, but numbered in the wrong direction. The full-inch scale on the bottom, left to right, should have diagonals in the left box, slanting bottom up to the left. But we observe that the diagonals in the left side box are slanted the wrong way around! One could argue that the diagonals are to be read top to bottom, but then the numbering of the horizontals is wrong.

When we look at Bion's picture of a double diagonal scale in the English translation by E. Stone [5], we see a comparable drawing error. This error, however, was introduced by Stone when he added the completely new Plate IV: the original French version, see [4], did not contain a double diagonal scale at all, only correctly drawn single diagonal scales in the plates *"Planche Quatrième"* and *"Planche Sisième"*.

Looking at actual specimens of Gunter rules, some show the same type of error in the slanting direction or numbering of the diagonals, hardly surprising when major handbooks on the subject give the wrong direction. As a specific example, the Gunter rule, drawn in [13] as an exact copy of an English-made "Potter Poultry" rule, has the same type of error. A total of 54 Gunter rules have been examined on this aspect, and five incorrectly drawn diagonal scales were encountered. Among these five, there were even two rules with all diagonals drawn exactly vertical.

Looking at a large number (more than 100) of actual architect's scaling rules, plotting rules and protractors with diagonals, no such errors have yet been detected. It appears that makers of drawing instruments paid more attention to the correct drawing of the diagonal scales than Gunter rule makers.

Non-linear scales

The principle of rectilinear diagonal scales is based on linear interpolation. However, examples of rules have been identified where straight diagonals were applied on non-linear scales like those for chords or sines (see Figure 8, part of a plain-scale in Museum Boerhaave, Leiden, where H stands for "Hoeckmaet" which is old Dutch for "sine"). In principle, the diagonals in that case should have been replaced by the vertical projection of the scale function involved, but the task of engraving curves instead of straight lines was apparently too difficult or costly. Anyway, the error introduced by such a linear interpolation for a sine scale does not exceed 0.5%.

Curved diagonals for the sine function did, however, exist, for example on the series of Pilot Balloon Slide Rules designed and produced for the UK Meteorological Office between 1915 and the 1970s, to convert observed azimuth and elevation readings of a 'meteo' balloon into wind velocity and direction (see Figure 9). This "diagonal" magnified the notorious imprecise area of the sine function between 70° and 90°.

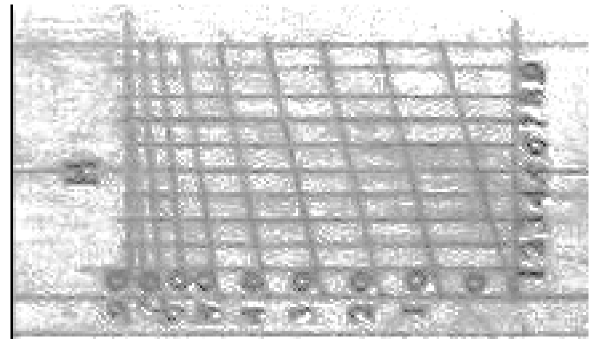


Figure 8. Sine scale with straight diagonals. (Museum Boerhaave, Leiden).

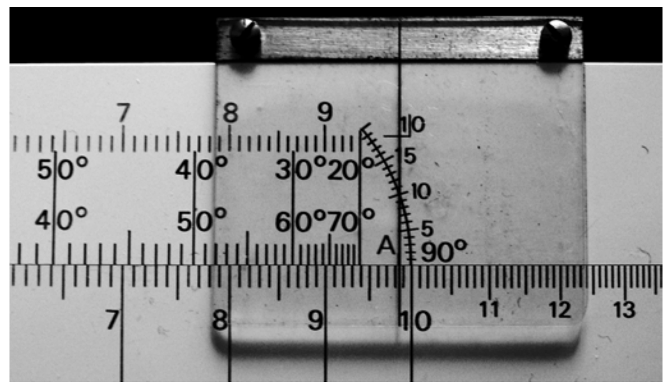


Figure 9. Curved sin/cos diagonal on the Blundell Mk-V Balloon Slide Rule showing under the hairline of the cursor: sine $(80°) = 0.985$

For logarithmic scales on a slide rule, a linear interpolation on straight diagonal scales would not be a very good idea, especially in the lower range where the error can exceed 2%.

Circular scales

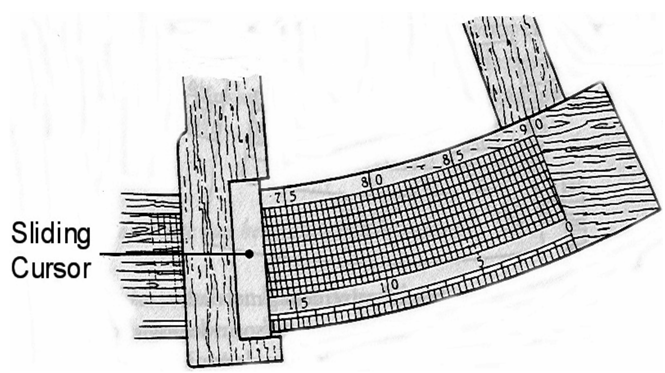


Figure 10. Diagonals on a circular scale with a precision of 2 minutes of arc, from [17].

Most instruments for measuring angles have circular scales, for example medieval astrolabes, 16^{th} century quadrants, 17^{th} century Davis quadrants (back-staffs), and later octants or sextants for navigation at sea. Diagonal scales with straight lines have been applied on circular scales to achieve sub-degree reading precision, see Figure 10.

An alternative implementation was to incorporate the diagonal line in the index. This approach is discussed in [14] for protractors.

In principle, the straight line diagonal on a circular scale should have been a weak "spiral" curve for correct linear interpolation, but the error in actual examples was negligible, see also [10].

Vernier scale

A major improvement in precise reading of scales with sliding cursors was realized by the moving Vernier scale which is still employed today, for example in calipers and sextants. Figure 11 gives the reading of a caliper, set to 98.2 units, with 98 on the stationary scale and the remaining fraction 0.2 on the sliding Vernier scale.

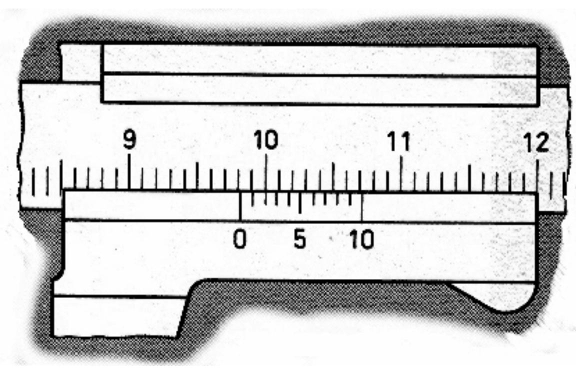


Figure 11. Vernier scale set to 98.2.

The sliding scale contains 10 units, each of which has a length of 9/10 of a unit on the fixed scale, so that at any position only one of the Vernier marks aligns with a unit mark on the fixed scale: and that particular Vernier mark gives the value of the last decimal of the reading.

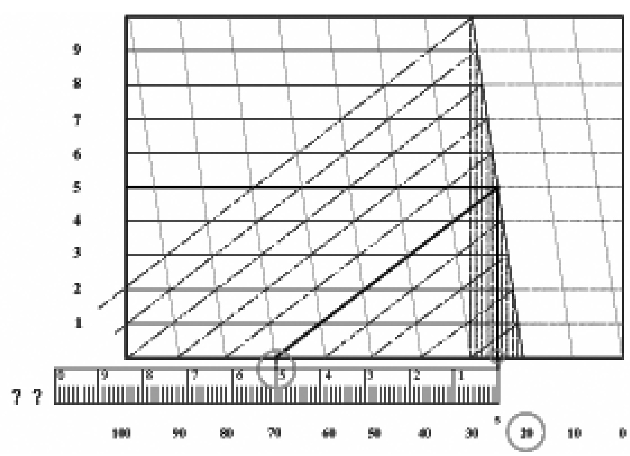


Figure 12. Vernier reading of 20 + 5 units.

The principle of the Vernier scale can be related to the diagonal scale as follows. In Figure 12 we look again at a measure of 25 units, measured from point zero. The two-dimensional diagonals are projected back onto the horizontal scale along the slanted lines, so that the five (unmarked) units on the horizontal scale between 20 and 25 are "magnified" to (10-5) + 10.(5-1) = 45 units on the horizontal scale between 25 and 70. The sliding Vernier scale (with its zero mark just below position 25 of the fixed horizontal scale) is divided in ten divisions of nine units each, so its division 5 exactly matches $5 \times 9 = 45$ units.

We can conclude that the Vernier effectively creates a glass-less magnifier, enlarging nine times in this example.

Many variations exist; for example, other magnifications can be realized by choosing a different number of units on the Vernier scale (20 is also very common). Non-decimal divisions are used for circular scales or other non-metric units. Sometimes the Vernier scale is duplicated (around the mark zero) to allow end-of-scale readings at both extreme ends. The scale precision of a modern handheld sextant with drum micrometer, moving Vernier scale and lens magnifier, can reach 5 seconds of arc. The Vernier is easier to use, compared to diagonal scales, and uses less lateral space. This method has been described—and still is—in a multitude of books and manuals on machine tooling, measuring, etc. (an example being again [9], page 135-137).

History of scales for enhanced precision

In a charming and entertaining French book, *Curiosités Géométriques* [8], a historical summary is given of diagonal, nonius and Vernier scales. Also Daumas [10] gives more background information.

The diagonal scale was probably used since medieval times, and has already been documented by the Jewish philosopher and astronomer Rabbi Levi ben Gershom (also known as Gersonides). In *Milhamoth Adonai* (Wars of the Lord, 1329, see [11]), he described an early cross-staff, "bacullus Jacobi" (see Figure 13), including transversals on the staff for more precise reading (although much of the precision must have been lost in the handling of this device).

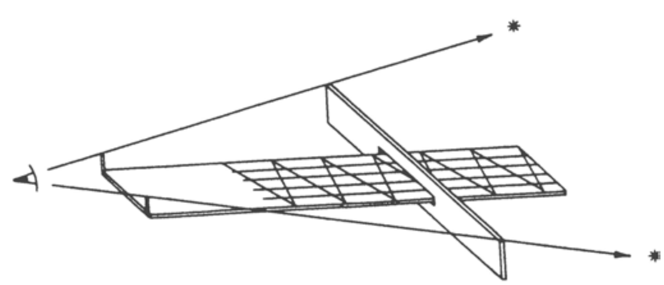


Figure 13. Gersonides' cross-staff with transversals (1329), from [11].

Gersonides also described circular scale transversals for use on the astrolabe, an example of which is depicted in Figure 14 (an astrolabe in the Florence "Istituto e Museo di Storia della Scienze", dated 1483).

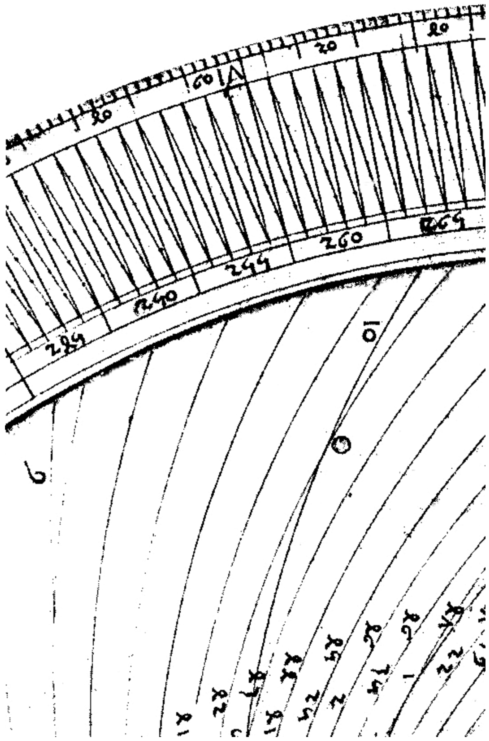


Figure 14. Transversals on astrolabe (1483), from [11].

In 1542, the famous Portuguese mathematician and navigator Pedro Nuñez (Petrus Nonius in Latin) designed a mechanism for precise circular scales, see [1]. On a standard scale of 90 degrees he added 44 concentric circles on positions 89p through 46p, each divided into a specific unit size: a scale unit on position n had an arc of $90/n$ degrees.

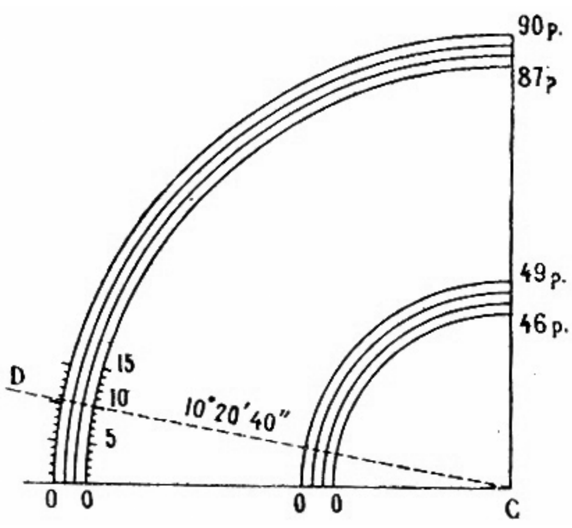


Figure 15. Original Nonius scales (1542), from [8].

The effect was that almost all arbitrary angles had at least one exact intersection with a scale unit on one of the 45 circles. Figure 15 gives the example where an unknown angle intersects unit 10 on circle 87p. On this circle one unit is an arc of 90/87 degrees, therefore ten units represent the given arc as $900/87 = 10°20'41''$. This set of scales is very difficult to make, and also inconvenient to use. Reference [16] gives more information about the original "Nonius" scale system, and identifies one quadrant (made by James Kynuyn, 1595) using this mechanism, in the Florence "Istituto e Museo di Storia della Scienze".

The Jesuit astronomer Christopher Clavius (who was the driving force behind the design of the Gregorian calendar) published in 1612 his own ideas about the scales of Nuñez and also laid the foundation for the Vernier scale, see [2].

In 1631 Pierre Vernier, mathematician and military engineer, published the implementation of his sliding scale, see [3], still in use today. In a number of countries (France, England, USA) the name "Vernier scale" is used, but there are also countries (like Germany, the Netherlands) where a Vernier scale is called a "nonius scale", which is not really correct, given this history.

Although technical drawings are now generally made with CAD (Computer Aided Design) computer programs, some drawing instruments (including diagonal scales) are still commercially available in mid-2004, for example from the German firm HAFF, see [18].

Acknowledgments

Discussions with Sigismond Kmiecik, Leo de Haan and Frans Vaes, and contributions from Jim Bready, have substantially contributed to this paper.

Many fellow-collectors have provided information on their diagonal scales, to help in building a picture of the number of "wrong" diagonals on Gunter rules.

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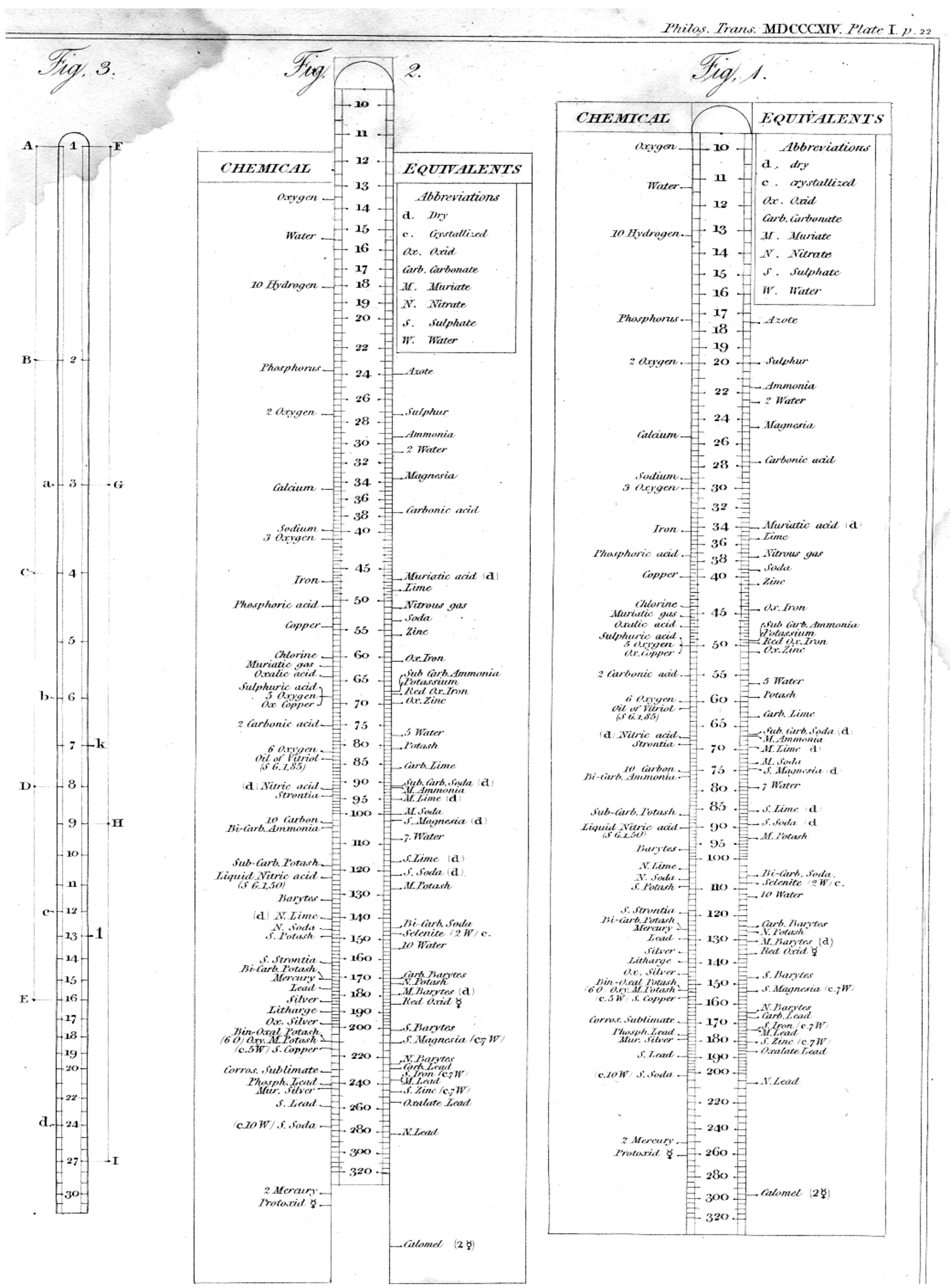
[17] W.F.J. Mörzer Bruyns, *Schip Recht door Zee. De octant in de Republiek in de achttiende eeuw*, Amsterdam, 2003, drawing on page 37 by N.H. Kamer

[18] Gebrüder HAFF GmbH, Fabrik für Zeichengeräte, Planimeter, Mathematische Instrumente. Transversalmaßstäbe: 250 mm lang, 40 mm breit, 2 mm stark, aus Hartmessing matt vernickelt, mit 4 Transversalteilungen (beide Seiten). Die Striche sind sehr fein, und genau geteilt und schwarz eingefärbt.[2]



A chemical slide rule. From an article by William Hide Wallaston, *Philosophical Transactions,* 1814.

[2]Copied on July 25, 2004 from *www.haff.de/transversalm.htm.*