Adaptation of Hemmi 153 Examples on Calculation of Hyperbolic Functions of Complex Arguments to the Flying Fish 1002 and 1003

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Background

Solving hyperbolic functions for complex arguments on a slide rule is a “really complex” task, since a two dimensional problem is solved by one dimensional calculations. For its solution on slide rules many different recipes exist, either directly calculating the real and imaginary part of the complex number or its amount and angle separately. For the latter approach again at least two different recipes are known. One of it which usually is not used makes use of the Pythagorean approach but not many slide rule models have appropriate scales other than A and B to use these approaches.

The Flying Fish (FF) slide rule models 1002 and 1003 comprise the Pythagorean H scales. The short manual for the FF1002 does show how to do basic vector calculations by means of H scales but not how to solve complex hyperbolic expressions such as sinh(a+jb) using these scales. The most likely first slide rule which manual propagated the use of this approach is the Hemmi 153. The Hemmi 153 comprises quadratic P, P' and Q scales which - though non logarithmic - share some ideas with the logarithmic H and P scales and also can be used for Pythagorean based vector calculations.

Therefor the focus here is, whether and how the calculation methods shown in the Hemmi 153 manual are applicable to FF1002 and 1003.

The Flying Fish model 1200 does not contain H scales. By means of B scale which can be used as a virtual H scale the recipes can be also applied on this model as on similar models with trig scales on the slide and the hyperbolic scale on the body.

Hyperbolic functions of complex arguments on the Flying Fish 1002 / 1003 as well as on FF1200

Some words on complex numbers

Complex numbers consist of a real part and an imaginary part. It may be assumed to be a point in an x-y coordinate system with x being the real part and y being the imaginary part. Complex numbers in its Cartesian or rectangular representation are usually written as a+jb, where a represents the real part (x-coordinate) and b represents the imaginary part (y-coordinate). Mathematicians use I instead of j. Electrical engineers however prefer j because I could be mixed up with an I standing for a current. Additional the j is more visible and hence it is simpler to distinguish real part and imaginary part with the gaze of the eye. When we assume the point (0,0) in the coordinate system being connected with the point (a, b) then this line has a certain length called the absolute amount, usually denoted as a capital character, and a certain angle measured from the x coordinated. This angle is often called phase angle or simply phase. If we assume now that the line has also an arrow at its end, then it is maybe clear why a complex number is also called a “phasor” or “vector”.

So a value a+jb can be also represented as $a+jb=A\angle\varphi$, with A being the absolute amount and φ the angle. The transfer between both representations is done according to transformations between polar and rectangular coordinate systems. The conversion functions are like this:

$$A=\sqrt{a^2+b^2};\varphi=\tan^{-1}\left(\frac{b}{a}\right);a=A\cdot\cos(\varphi);b=A\cdot\sin(\varphi)$$

Recipes on the Hemmi 153

The basic recipe of the Hemmi 153 to calculate sinh of complex numbers is denoted as:

$$\sinh(a+jb)=\sqrt{(\sinh^2(a)+\sin^2(b))}\angle\tan^{-1}\left(\frac{\tan(b)}{\tanh(a)}\right):$$

In a similar way cosh is defined:

$$\cosh(a+jb)=\sqrt{(\sinh^2(a)+\cos^2(b))}\angle\tan^{-1}(\tan(b)\cdot\tanh(a))$$

The manuals of most vector slide rules offer (beside others) this definition of tanh:

$$\tanh(a+jb)=\frac{\sinh(a+jb)}{\cosh(a+jb)}=\frac{|\sinh(a+jb)|}{|\cosh(a+jb)|}\angle(\varphi_{\sinh}-\varphi_{\cosh})$$

The Hemmi 153 offers a slightly different definition:

$$\tanh(a+jb)=\frac{\sqrt{\sinh(a)^2+\sin(b)^2}}{\sqrt{\sinh(a)^2+\cos(b)^2}}\angle\tan^{-1}\left(\frac{\sin(2b)}{\sinh(2a)}\right)$$

While the absolute amount of the complex number is given identically and meant to be calculated separately before a simple division, the angle can be calculated in one simple step, if the doubling is done in the head.

Rad deg conversion

Usually the imaginary part b is given in rad and must be converted to deg first to calculate the sin and cos of this angle on the Flying Fishes. In order to e.g. convert 1.22 rad into degree and vice versa on the Flying Fish 1002 its ticks mark 1° and R are used which correspond to pi/180 respectively 180/pi:

C 1° to 1.22 on D; at C10 69.9deg on D

Converting back is possible by

R on C to 69.9 on D; at C1: 1.22 rad

Other conversion methods are possible be using the two methods “backwards”.

On a FF1003, the FF1200 or another slide rule with ST scale (being assumed on slide):

At 1.22 on C: 69.9 on ST.

If reading accuracy on ST is not sufficient, 1° on ST can be used similar to the tick mark 1° on the FF1002:

ST 1° to 1.22 on D; at C10 69.9deg on D

Converting back may be done by using this method “backwards”:

C10 to 69.9deg on D; at ST 1°: 1.22 on D

sinh(a+jb)

The example used is sinh(0.32+j1.22) = sinh(0.32+j69.9°).

Before showing the actual recipe some theoretical background is presented how to derive the recipes.

Since the H scales follow the pattern sqrt(1+(0.1c)^2), here denoted H1, and sqrt(1+c^2), here denoted H2 either $\sinh(a)$ or $\sin(b)$ must be factored out, in order to calculate the absolute value of the vector:

$$\sqrt{(\sinh^2(a)+\sin^2(b))}=\sinh(a)\sqrt{1+\left(\frac{\sin(b)}{\sinh(a)}\right)^2}$$ or

$$\sqrt{(\sinh^2(a)+\sin^2(b))}=\sin(b)\sqrt{1+\left(\frac{\sinh(a)}{\sin(b)}\right)^2}$$

In both equations it is essential to know the value range of the fraction inside the root, so that the appropriate H scale is chosen. Otherwise there is no need to write down the intermediate values. sinh(0.32)=0.3239 and sin(69.9°)=0.938

On the Flying Fishes in scope and many others the trig scales are on slide while the hyperbolic scales are on body. Further the H scales of FF1002 and FF 1003 are on the slide. Thus in order to use the H scales, the intermediate result for the fraction under the root must be visible on C scale.

In general there are two ways to perform divisions on the slide rule by non inverted scales which yields the result on C: Either simply “inverting” the standard method of division by C and D or to use both values on D. Calculating e.g. 2/3 by inverting the standard method:

2 on C to 3 on D; at 10 on D: 0.66 on C

When calculating 3 by 2 by means of the standard method at the index of C 1.5 is visible on D. In general the slide rule at the index of D always the reciprocal value of the index of C.

Using the method of using the values on D:

10 on C to 3 on D. At 2 on D: 0.66 on C.

Please note that the index of C is set to the divisor on D so that with the same slide setting we can multiply the divisor and this is exactly what we need according to factoring out the divisor in the equations above.

Last but not least the sin scale is on slide so that it is necessary to start with the sin value first, because after setting the slide there is no opportunity anymore to obtain the sin of a value on D scale. Therefor we need to factor out sin(B) and need to take into account that in this example sinh(0.32)/ sin(69.9°) is smaller than 1.

Therefor the supposedly most efficient method on 1002 is the following:

C1 to D1. Cursor to 69.9 on sin

Remark: by this we find the position of sin(69.9) on D

C10 to cursor; at 0.32 on sinh: 1.05825 on H1

Remark sinh(0.32) < sin(69.9), therefor we use the right index of C

At 1.05825 on CF: Absolute amount of the vector 0.9925 on DF

In order to calculate the angle of the vector by $\tan^{-1}$ again it is important to take into account that the scale is on slide and therefor the result of tan(b)/tanh(a) needs to be on slide respectively C. This can be easily achived by means of the “inverted” normal division.

This leads to the supposedly most efficient method on 1002:

69.9 on tan to 0.32 on tanh; at D10: 83.53 on tan

Finally the angle can be back-transferred to rad:

R on C to 83.53 on D. At C1: 1.461 on D

cosh(a+jb)

For cosh the example cosh(0.257+j0.652) is used.
0.652 rad converts into 37.4°.

sinh(0.257)=0.26 and cos(37.4°)=0.795.

The calculation of the absolute amount of the vector follows the same pattern as for sinh with except that cos is used instead of sin.

C10 to D10. Cursor to 37.4° on cos.
C10 to cursor because cos(37.4°)>sinh(0.257)
At 0.257 on Sinh: C0.337 and H1 1.052
At 1.052 on CF: 0.836 on DF

For the calculation of the angle we need to take into account that this time a multiplication of two values on non inverted scales is needed and that the result needs to be available on C again. The easiest way to perform this is to “invert” the standard multiplication by means of C and D. E.g. 2*3:

2 on C to 1 on D; at 3 on D: 6 on C

tan(37.4°) = 0.764 >tanh(0.257) = 0.2515

37.4° on tan to D10; at 0.257 on tanh: 0.192 on C and 10.88° on tan

Back conversion to rad:

R on C to 10.88 on D; at C10: 0.19 on D

tanh(a+jb)

The amount of the complex number may be written as:

$$|\tanh(a+jb)| = \frac{\sqrt{(\sinh^2(a)+\sin^2(b))}}{\sqrt{(\sinh^2(a)+\cos^2(b))}} = \frac{\sin(b)\sqrt{1+\left(\frac{\sinh(a)}{\sin(b)}\right)^2}}{\cos(b)\sqrt{1+\left(\frac{\sinh(a)}{\cos(b)}\right)^2}}$$

Using the example tanh(1.25+j0.28) the first is again to convert 0.28 rad into 16.05° by:

1°on C or ST to 0.28 on D; at C1: 16.05°

The following steps are as known from sinh and cosh and it may be appropriate to start with cosh and to write down this intermediate result:

C10 to D10; cursor to 16.05 on Cos

C10 to cursor; at 1.25 on sinh: 1.66 on C, 1.938 on H

At 1.938 on C: 1.868 on D (write down intermediate result)

C1 to D1; cursor to 16.05 on sin

C10 to cursor; at 1.25 on sinh: 5.78 on C, 5.87 on H

At 5.87 on C: 1.672 on D

Look for the intermediate result but don’t change the cursor yet!

1.868 on C to 1.672 on D; At C10: 0.87 on D

The amount of the complex number also calls for this reformulation:

$$|\tanh(a+jb)| = \frac{\sqrt{(\sinh^2(a)+\sin^2(b))}}{\sqrt{(\sinh^2(a)+\cos^2(b))}} = \frac{\sinh(a)\sqrt{1+\left(\frac{\sin(b)}{\sinh(a)}\right)^2}}{\sinh(a)\sqrt{1+\left(\frac{\cos(b)}{\sinh(a)}\right)^2}} = \frac{\sqrt{1+\left(\frac{\sin(b)}{\sinh(a)}\right)^2}}{\sqrt{1+\left(\frac{\cos(b)}{\sinh(a)}\right)^2}}$$

This may be calculated by

16.05 on cos to 1.25 on sinh: at 1 on D C: 0.6, H:1.166 write down 1.166

16.05 on sin to 1.25 on sinh: at 1 on D C: 0.1725, H:1.0148

C1.166 to D1.0148; at C10: 0.871 on D

This approach is faster, since the initial reset of the slide to the index is not needed as well as the multiplication with the term factored out.

The original Hemmi 153 formulation for the angle maybe very quick if doubling is done in the head.

$$\angle(\tanh(a+jb))=\tan^{-1}\left(\frac{\sin(2b)}{\sinh(2a)}\right)$$

sin 32.1 to sinh 2.5: C0.0878

Attention! Since 0.0878<0.1 the tangent of an angle is approximately its value in rad. Therefor we already have the result, which we can convert back into deg if necessary:

1° on C to 0.0878 on D; at 1 in C: 5.04°

On a FF1003 the result could be directly read in degree on the ST scale.

The second possibility is to calculate the angles for sinh and cosh and to calculate their difference.

tan 16.05 to tanh 1.25; at D10: C0.339 at tan: 18.71°

tan 16.05 to D10; at tanh 1.25: C0.244; tan 13.71°

The result is then 18.71°- 13.71° = 5°

Also this could be calculated on the slide rule:

C1 to 13.71 on D; At D18.71: C1.365. Substruct 1 in the head: 1.365-1=0.365

At C 0.365: 5° on D.

Transfer on the FF 1200

The FF1200 does not comprise H scales. Therefor the use of H scale must be replaced by B scale. Hence only the calculation of the amount is affected, but not of the angle.

|sinh(0.32+j1.22)| = |sinh(0.32+j69.9°)|:

C1 to D1; Cursor to 69.9 on sin
C10 to cursor; at sinh 0,32: C0.347 B: 0.121 add 1 in Head: 1.121
Cursor to 1.106 on B. Problem: no scale extension on D; thus:
C1 to cursor. At C10: 0.995 on D

|cosh(0.257+j0.652)| = |cosh(0.257+j37.4°)|:

C1 to D1; Cursor to 37.4 on cos
C10 to cursor; at sinh 0.257: C0.327 B: 0.107 add 1 in Head: 1.107
Cursor to 1.107 on B. Attention: not enough space for the cursor.
Therfor cursor to C10; C1 to cursor

Cursor to 1.107 on B. At D: 0.832

|tanh(1.25+j0.28)| = |tanh(1.25+j16.05°)| (Quick solution, otherwise like sinh and cosh):

16.05 on cos to 1.25 on sinh; At C: 0.6; at B:0.36. Add 1 in head: 1.36
1.36 on B out of scale therefor first: Cursor to C10; C1 to cursor
At 1.36 on B: 1.165 on C (write down!)

16.05 on sin to 1.25 on sinh; At C: 0.173; at B:0.03. Add 1 in head: 1.03
1.03 on B out of scale therefor first: Cursor to C10; C1 to cursor
At 1.03 on B: 1.018 on C

1.165 on C to 1.018 on D; at C10: 0.87

Since without the H scale, the folded scales CF and DF can't be used set backs are often required. Otherwise the "virtual H scale" method is a quite performant method which only needs in the case of tanh two cursor settings more than the usage of the H scale in order to read the result on C scale.

Summary

It was possible to transfer the calculation methods of the Hemmi 153 to the Flying Fishes 1002. 1003 and 1200.

The H scales following the pattern sqrt(1+c^2) and sqrt(1+(0.1c)^2) requires to factor out one of the factors. This may be regarded as a disadvantage seen from the point that the trig value, either sin or cos, needs to be set on D which is only possible by a first alignment of the indices of C and D.

For calculation of the absolute amount of tanh(a+jb) a method can be derived, which avoids factoring out and therefor of high calculation speed. This is of a certain relevance since the tanh is most likely the most necessary hyperbolic function in calculations of transmission lines.

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