NOTE: This is a revised and expanded version of my original Article on *Slide Rules With Hyperbolic Functions,* and an updated version of my *Listing of Slide Rules with Hyperbolic Scales*. These first were published in The Journal of the Oughtred Society, Vol. 14, No. 1, 2005.
The revised Article and Listing are shown as separate parts on this Web Site. The first part is the Article that immediately follows. Then the second part, the *Listing of Slide Rules with Hyperbolic Scales*, is found by returning to the main menu.

Slide Rules with Hyperbolic Function Scales

By William K. Robinson

Introduction

The introduction in late 1929 of the K&E 4093-3 Log Log Vector slide rule with hyperbolic scales should be considered a major milestone in slide rule history. This is not for the reason that one could now directly read the values of the hyperbolic functions for the first time on a slide rule - but more for the reason that one could now readily calculate the values of the complex hyperbolic functions. This new slide rule provided rapid solutions for complex expressions such as $\sinh (u + j\,\theta) = A\,/\underline{\alpha}$, in the Polar form ; or, $\sinh (u + j\,\theta) = (x + j\,y)$, in the Cartesian form. These complex functions were being encountered more and more often in electrical engineering and other scientific applications of the time. Prior to the advent of this slide rule these complex hyperbolic equations were handled by a lengthy hand calculation routine using logarithms. We will see later in this Article how this breakthrough in slide rule design by Mendell P. Weinbach gave the engineer an immensely valuable and timesaving tool. Its importance really cannot be overstated.

Slide rules with hyperbolic function scales are of major interest to me because I used this type of slide rule in my first real job. This was as an engineer in the Vibration and Flutter Unit of Boeing in Seattle, beginning in the early summer of 1946. The job was full time in the summer and part time the rest of each year as I finished my University studies. My work was mostly involved in solving the various flutter modes of the B-47 and B-52 bombers by matrix iteration. Other work was in the wind tunnel and in original research involving vector solutions of the flutter matrix. For one work assignment I designed a slide rule, six inches wide and two feet long, to sort out the various vibration modes coming from the print outs of our stress strain gauge tests.

My beginning job title in 1946 was “Vibration and Flutter Computer”. This was a perfect description as I really was a human computer. A large part of my early tasks included endless complex number calculations, done by hand on a desk calculator. These were lengthy iterations calculated to solve different versions of the flutter matrix. Many solutions took months of time to complete. About a year and a half after starting work I was promoted with the title of Engineer and relieved of what seemed like a never ending routine.

Overall this was a fascinating cutting edge job, and I often used my K&E 4083-3 Log Log Duplex Vector slide rule. It was always at my side – and for the hundred or more Engineers around me their slide rules were always beside them in plain sight on their drafting table or desk. My K&E 4083-3 was an unusual model to have at the time as most engineers were trained in the use of and carried a K&E 4080-3, K&E 4081-3, or earlier model. (Remember, the Dietzgen and Pickett slide rules did not emerge in the market until after WWII, and those with hyperbolic scales did not appear until about 1948. So, all I recall seeing in my early years at school and work were K&E rules).

One should understand that in 1946 what we know today as calculators and computers were many years away into the future. Also, it was a few years before IBM punch card calculating machines would arrive for our use in solving problems. For me these early years were the *Golden Era* of the slide rule. It was the main tool being used by us for building our very advanced swept-wing B-47 and B-52 bombers. And if one stopped to think about it, it had been the main tool used up to that time throughout the world in the design and construction of all achievements in science and engineering. (This is a long and impressive list including all buildings, bridges, dams, automobiles, trains, airplanes, manufacturing plants, laboratories, and etcetera). And, as we know, the slide rule would be used for many more years until gradually replaced by IBM machines, computers, and finally in the 1970's by the hand calculator.

The resources we engineers had to work with in 1946 were: (1). the slide rule; (2). tables of logarithms; (3). tables of the different math functions and their logs; and, (4). various specialized tables, graphs, charts, and nomograms. These other sources (2)-(4) were consulted when more accuracy or other answers were needed that could not easily be provided by using the slide rule. Although the slide rule was by far the main tool we used, I feel these other sources should be mentioned in addition to the slide rule, as the availability of these for our use when needed was important.

We did have one advantage in 1946 that engineers before World War II did not have. This was the use of the newly introduced "desk calculating machines". As they were just becoming available in 1946 it was rare to see these anywhere else at Boeing, or in fact in any office. However, we had three in our Vibration and Flutter Unit for the reason that our work was involved in very high level mathematics. And the machines were needed and there for one purpose - to save calculating time in completing the weeks and weeks of iterations needed to solve the flutter matrix. There were two Marchant's and one Friden desk calculating machine in our Unit. We used them mainly for our complex number calculations. Also, these were used for regular multiplication and division when more accuracy was needed to solve our vibration, flutter, and stress analysis problems. (These machines would be called real clunkers by today's standards, but were state of the art at the time). The desk calculators had no paper print outs, so every answer was copied by hand. In fact, every thing we calculated in those years had to be recorded by hand. What we know today as machines with print outs were years away.

I was very happy to have my K&E 4083-3 Log Log Duplex Vector slide rule to use due to the kind of advanced complex mathematical problems we encountered in my Unit. The two Slide rules with Hyperbolic Scales (the K&E 4093-3 in 1929; and the K&E 4083-3 in 1939) had become important engineering tools by the time I went to work at Boeing. The reason for this is that hyperbolic functions are encountered in many areas by scientists, engineers, physicists and mathematicians. They are found in a very wide range of applications, from transmission lines to Einstein's theory of relativity. Growth for their use had blossomed since the early decades of the 1900's when industrial inventions employing formulas with hyperbolic functions were introduced. Continued expansion of electronic, and other scientific, applications of hyperbolic functions had further pushed the need for these slide rules forward.

A Short History of Hyperbolic Functions

It may be helpful to know something about the origins of hyperbolic functions to better understand our slide rules. This short history has been adopted mainly from *Hyperbolic Functions (Smithsonian Mathematical Tables),* by George F. Becker and C. E. Van Orstand, 1909 (Second reprint 1920). I reviewed a few other references on the history of mathematics to confirm the important dates and people involved. Two of these that the reader would enjoy were; *e: The Story of a Number*, by Eli Maor, 1994, and, *An Imaginary Tale The Story of √-1*, by Paul J. Nahin, 1998.

Hyperbolic functions were not introduced until around the late 1750's. However, it was almost two hundred years earlier that the first and one of the most important applications of the functions now known as *hyperbolic* was made by Gerhard Kremer (or Krämer), 1512-1594. He was the Flemish geographer who was better known by his Latin name, Gerhardus Mercator. In 1569 he issued his Mercator's Projection map. His projection resulted in the making of a map in which a straight line (*the loxodrome*) always made an equal angle with every meridian. This was a significant and major breakthrough in navigation. Its importance is evidenced by the fact that today all deep-sea navigation charts of the world have as their basis this projection. Mercator published his map without explanation, and it was left to others following him to discover that the formulas he used were linked to the hyperbolic functions. We will show the details of his formulas a little later in this Article.

The development of the mathematics of hyperbolic functions emerged in the 1700's and 1800's through the contributions of the following pioneers:

Vincenzo Riccati (1707-1775) is noted as the actual inventor of hyperbolic trigonometry. In 1757 he introduced the use of hyperbolic functions to obtain the roots of certain types of equations, particularly cubic equations. He adopted the notation Sh.ϕ and Ch.ϕ for the hyperbolic functions, and Sc.ϕ and Cc.ϕ for the circular ones.

Soon after, in 1759, Daviet de Foncenex (1734-1799) showed how to interchange circular and hyperbolic functions by using the $\sqrt{-1}$. This was made possible from the earlier contributions of other mathematicians. Such as De Moivre's (1667-1754) classical equation of 1722, $(\cos x + i \sin x)^n = (\cos nx + i \sin nx)$. This was followed in 1748 by the well known *Euler's Equation*, $e^{\pm ix} = (\cos x \pm i \sin x)$. From these various efforts emerged the now familiar links between the circular and hyperbolic functions of:

$$\sin \alpha = - i \sinh i\alpha, \text{ and } \cos \alpha = \cosh i\alpha;$$

and the converse equations: $\sinh \beta = - i \sin i\beta$, and $\cosh \beta = \cos i\beta$.

Using these identities it was found that many substitutions could be readily made between the circular and hyperbolic functions.

We find that the first systematic development of hyperbolic functions was by Johann Heinrich Lambert (1728-1777). In a 1768 paper he adopted the notation we use today; *sinh u, cosh u,* etc., and introduced what he called was the *transcendent angle,* using it in computation and in the construction of tables (later this was renamed the *gudermannian*). He is credited with popularizing the new hyperbolic trigonometry that modern science finds so useful. It has been said that Lambert did for hyperbolic functions what Leonard Euler (1707-1783) had done for circular functions.

Of historical interest is that it was Euler who introduced both the mathematical symbols *"e"* in 1728, and *"i" for "√–1"* in 1777. Printers were very happy to see this new *"i"* symbol as it made their job easier. The *"√–1"* had not only been an awkward symbol for authors to write, but also difficult for the printers to insert in formulas and equations when typesetting them. Good examples of this can be found by looking at the formulas and equations above on the previous page. When writing these I used the symbol *"i"* instead of *"√–1"*. However, in all of these the symbol *"√–1"* should have been used. The reason for this is that the original writings by the various authors were all made before 1777. So, the *"√–1"* was the only symbol they, and the printers, then had available to enter in the formulas, and this was clumsy to use.

Later in 1832, Christoph Gudermann (1798-1852) published an important paper followed by extended tables of the hyperbolic functions. These were based on the solution of Mercator's projections that had been found to be: $\lambda = gd$ (m/a), and (m/a) = ln tan [(π/4) + (λ/2)], where λ is the latitude, m is the projection point in latitude λ, and a is the radius of the Earth. The term *"gd"* is called the *gudermannian* after Gudermann, who introduced the term. The general equations for the *gd* follow those found for Mercator's formulas. These are:

If, $x = \ln \tan [(\pi/4) + (\theta/2)]$, then $\theta = gd\ x$ = the *gudermannian* of x. Of interest is that the *gudermannian (gd)* provides an important linkage between the circular and hyperbolic functions. These are the following:

$\tan gd\ x = \sinh x$; $\sec gd\ x = \cosh x$; and $\sin gd\ x = \tanh x$.

These formula links provided Gudermann the means to calculate his tables. Years later, in 1862, Cayley, in recognition of Gudermann's significant contributions, proposed the name "*gudermannian*" for the angle that Lambert called "*transcendent*". The name, *gudermannian*, remains today, and the formulas and tables of values are shown in many Mathematical and Engineering Handbooks.

Modern interest in hyperbolic functions was accelerated with the invention and commercialization of electricity. The widespread use of electricity started with the telegraph's development in the mid 1850's, and then was followed by the telephone, electric light, and the fast growing need of industry for power. Generating plants and transmission lines started to cross the American continent. The transmission of electrical power involves the application of hyperbolic functions.

A new academic degree called *Electrical Engineer* was the Universities' answer to the need of the business world to handle electricity on a large scale. Impetus in the use of hyperbolic functions was increased in 1884 when the A.I.E.E. (*The American Institute of Electrical Engineers*) was founded. Starting in the late 1800's more and more uses of hyperbolic functions were found as the uses of electricity expanded.

As the academic world took notice of this growth it was aided by the publication of text books and more extensive tables of hyperbolic functions. Some of the important English textbooks of the time were those of Prof. James McMahon, *Hyperbolic Functions* (Mathematical Monographs No. 4) (New York, 1896), and Prof. A. E. Kennelly, *Hyperbolic Functions Applied to Electrical Engineering* (Harvard, 1911). The more well known tables were: Becker and Van Orstrand, *Hyperbolic Functions* (*Smithsonian Mathematical Tables*) (Washington, D.C., 1909); A. E. Kennelly, *Tables of Complex Hyperbolic and Circular Functions* (Harvard, 1914); J. B. Dale, *Five figure Tables of Mathematical Functions* (Ainold, 1918}; K. Hayashi, *Fünfstellige Tafeln der Kreis- und Hyperbel-funktionen* (Berlin, 1930); E. Jahnlie and P. Emde, *Funktionentafeln mit Formeln und Kurven* (German and English, Leipzig, 1933).

These were supplemented by books with nomograms, by which the complex hyperbolic functions could be found. Two of these were: A. E. Kennelly, *Chart Atlas of Complex Hyperbolic and Circular Functions* (Harvard, 1924); and, L. F. Woodruff, *Complex Hyperbolic Function Charts, Elec. Eng.,* Vol. 54, 1935. The Woodruff charts covered the solution of 60-cycle lines up to 300 miles long.

In the early decades of the 1900's, with the help of these tables and books, the applications of hyperbolic functions spread into every scientific discipline. Then the slide rule with hyperbolic scales appeared on the scene in 1929. We will see later in this article how important the slide rule became in solving and checking calculations involving both hyperbolic and complex hyperbolic functions.

The Mathematics of Hyperbolic Functions

Hyperbolic functions can be derived mathematically in a number of ways. The three usual approaches are: (1) the development by means of graphics, (2) the solution by differential equation, and (3) the solution by infinite series. In the section that follows we will show how hyperbolic functions arise from each of these three sources.

We will begin with the graphical approach. The study of hyperbolic functions using this method probably began early in the history of the Calculus when it was noticed that the area under the circle was the integral $\int \sqrt{(a^2 - x^2)}$, whereas the area under the hyperbola was the integral $\int \sqrt{(x^2 - a^2)}$. We note these two equations only differ by the signs of a and x. As the area under the circle can been obtained by using trigonometric functions it was thought that there might be some similar functional relation based on the area under the hyperbola. In addition, when the axis was rotated ninety degrees the area under the hyperbola ($y = b/x$) had been found to be related to the natural logarithm function. In fact, the early name for *natural logarithms* was *hyperbolic logarithms*. Over time these led some to think that there might be a number of relations, perhaps involving the trigonometric functions, the logarithmic functions, and imaginary numbers – and, of course, these ideas opened new windows and new paths to explore.

As expected, using a graphical approach, an analogy was found between the circular (trigonometric) and hyperbolic functions. We develop this by starting first with the circle. Consider θ as an angle forming a circular sector. (See the shaded area *MOP* in Figure 1 below). Now the area C of this circular sector *MOP* is ½ θ . Then twice C (the area of the circular sector *MOP*) is equal to the number of *circular radians* in the measure of the angle θ. Or, angle $\theta = 2C$.

For the unit circle $x^2+y^2 = 1$, where OM = 1, we find that: $\sin \theta = y/OM = y$, and $\cos \theta = x/OM = x$.

We can now develop an analogous statement for the hyperbolic functions. A picture of the hyperbolic sector *MOP* is found in the shaded area of Figure 2 below. This graph is of the unit rectangular (equilateral) hyperbola of $(x^2 – y^2) = 1$, or $y = \sqrt{(x^2 - 1}$.

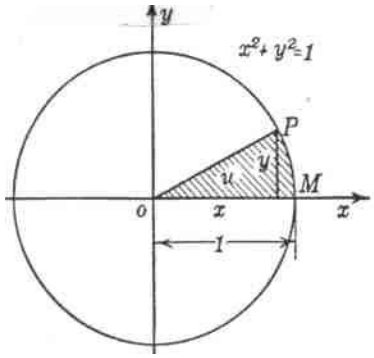


Figure 1.

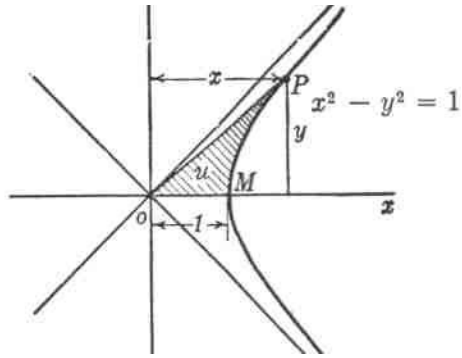


Figure 2,

In Figure 2, where *OM* =1, we draw a radius vector to a point P(x,y). Let H equal the area *MOP.* Then twice H (the area of the hyperbolic sector *MOP*) is equal to the number of *hyperbolic radians* in the measure of the angle *u.* Or, angle *u* = 2H.

Now, from Figure 2. we see that H, the area of the hyperbolic sector *MOP,* is the area of *xOP* less the area *xMP*; where area *xOP*= ½ xy, and area *xMP*= ∫ y dx, (from 1 to x). The mathematics to express H as twice the area MOP in terms of *u* follows:

u = 2H (twice the area of MOP) = 2 [½ xy – ∫ y dx, (from 1 to x)].

$u = [\, xy - \int \sqrt{(x^2 - 1)}\, dx$, (from 1 to x)],

$u = [\, xy - x\sqrt{(x^2 - 1)} + \ln(x + \sqrt{(x^2 - 1)})\,]$, (from 1 to x),

$u = [\, xy - xy + \ln(x + \sqrt{(x^2 - 1)}\,] = \ln(x + \sqrt{(x^2 - 1)}$.

From this $(x + \sqrt{(x^2 - 1)} = e^u$, and since $\sqrt{(x^2 - 1)} = y$, we can write,
$(x + \sqrt{(x^2 - 1)} = (x + y) = e^u$, and further that $[1 / (x + \sqrt{(x^2 - 1)} = (x - y) = e^{-u}$
Then, by subtracting $(x - y) = e^{-u}$ from $(x + y) = e^u$ we obtain $y = (e^u - e^{-u})/2$. Further, by adding $(x + y) = e^u$ to $(x - y) = e^{-u}$ we obtain $x = (e^u + e^{-u})/2$. These last two expressions are the familiar formulas that define sinh u and cosh u. So, for x and y we have:

$y = \sinh u = (e^u - e^{-u})/2$, and $x = \cosh u = (e^u + e^{-u})/2$

To complete our development of hyperbolic functions by the graphical approach we again refer to Figure 2. For the unit hyperbola we then proceed to write the following: sinh u = y/OM = y, and cosh u = x/OM = x. (Note these equations are similar to the circular functions, of $\sin\theta = y$, and $\cos\theta = x$).

As a last comment we need to point out that graphically it is not possible to draw the *hyperbolic angle u* in the same way that the *circular angle* θ is drawn. For *u* has no such reality. It is only exists as a function of the hyperbolic sector area H. It is important to avoid attempting to interpret *u* as an angle meeting at a point on the hyperbola.

The second illustration of how hyperbolic functions arise from mathematical sources is from the solution of a particular type of second order differential equations. For example, the solution of

$dy^2/dx^2 = k^2 x$, is $x = (a\cdot e^{kt} + b\cdot e^{-kt})$. This equation is converted to a solution in hyperbolic functions by using the known identities: $(a\cdot e^{kt}) = a\cdot(\sinh kt + \cosh kt)$, and, $(b\cdot e^{-kt}) = -\, b\cdot(\sinh kt + \cosh kt)$. So, by subtraction $(a\cdot e^{kt} - b\cdot e^{-kt}) = (a - b)\sinh kt$; and by addition $(a\cdot e^{kt} + b\cdot e^{-kt}) = (a + b)\cosh kt$,

or, $x = A \sinh kt + B \cosh kt$, where $A = a - b$ and $B = a + b$

An example employing this method of developing hyperbolic functions from this type of differential equation will be shown later in this Article.

The third illustration of the derivation of hyperbolic functions is from the infinite series for e^x using different values of x. It was Euler in his *Introductio in Analysin Infinitorum* (1748) who developed the infinite power series for e^x from the limit formula $\lim (n \to\infty) (1+x/n)^n$; then, $e^x = \lim (n \to\infty) (1+x/n)^n = 1 + x/1! + x^2/2! + x^3/3! + \dots$. From this infinite power series we can obtain the series for cosh x and sinh x. If we add the power series for e^x to that of e^{-x} , and divide by 2 , we can derive the power series for cosh x as follows:

$(e^x + e^{-x}) / 2 = [(1 + x/1! + x^2/2! + x^3/3! + \dots) + (1 - x/1! + x^2/2! - x^3/3! + \dots)] / 2$

So, $(e^x + e^{-x}) / 2 = (1 + x^2/2! + x^4/4! \dots + x^{2n}/ 2n! + \dots) = \cosh x$.

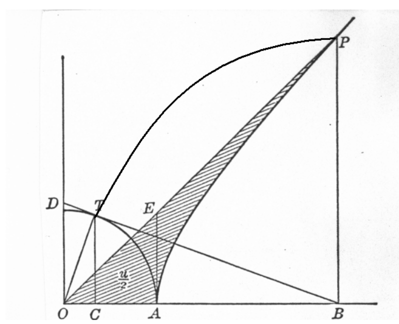
Similarly, if we subtract the power series for e^{-x} from that of e^x , and divide by 2, we can derive the power series for sinh x as follows:

$(e^x - e^{-x}) / 2 = [(1 + x/1! + x^2/2! + x^3/3! + \dots) - (1 - x/1! + x^2/2! - x^3/3! + \dots)] / 2$

So, $(e^x - e^{-x}) / 2 = (x/1! + x^3/3! + x^5/5! \dots + x^{2n+1}/ (2n+1)! + \dots) = \sinh x$.

Again, we have to thank Euler for his contributions to infinite series.

An Interesting Graphical Depiction of Hyperbolic Functions



An interesting demonstration showing all of the six hyperbolic functions by graphics is found in *Advanced Mathematics for Engineers*, by Reddick and Miller (page 89). We start by drawing a quadrant of a unit circle of radius OA =1. Then from point A to point P we construct a unit hyperbola $(x^2-y^2) = 1$. Then any point (x, y) on arc AP on the hyperbola is defined as x = OB and y = BP. Next, we take point B as a center, and BP as a radius, and draw a circular arc from P to intersect at point T on the unit circle. Then we draw line BT which is tangent to the unit circle at point T. This makes angle OTB a right angle. Also, since BP = BT, we know by definition that both BP and BT = y = sinh u.

Now taking the right triangle OTB we have OT = 1 and BT = sinh *u*. This means, $OT^2 + \sinh^2 = OB^2$, or $1 + \sinh^2 = OB^2$. Now we know that $1 + \sinh^2 = \cosh^2$, and also that the line OB = x. So, from this we have $OB^2 = \cosh^2 = x^2$, and so cosh *u* = OB = x. Then from BP / OB we have sinh *u* / cosh *u* = tanh *u*. Taking similar right triangles we can show that BP / OB = AE / OA = tanh *u*, and since OA = 1 this gives AE = tanh *u*.

We have now developed three of the six hyperbolic functions; sinh *u*, cosh *u*, and *tanh u*. Looking at our Figure 2 we see right triangles OTD, OTC and OTB. From these triangles we can compare some similar sides that will allow us to find the last three hyperbolic functions sech *u,* csch *u*, and coth *u*.

From the similar triangles OTC and OTB we obtain sech u as follows:

OC/OT = OT/OB, and then OC = 1/OB = 1/ cosh *u* = sech *u.*

From the similar triangles OTD and OTB we obtain csch u and coth u as follows:

DT/OT = OT/BT, and then DT = 1/BT = 1/ sinh u = csch u; and

OD/OT = OB/BT, and OD = OB/BT = cosh u / sinh u = coth u.

We can now summarize the graphical representations of the six hyperbolic functions:
BP = sinh u, OB = cosh u, AE = tanh u, OC = sech u, DT = csch u, and OD = coth u. This is a remarkable development of these functions. However, there is still one more unusual function of u to show. This is the *gudermannian of u* that is found from the angle AOT. So, $gd\ u = \angle AOT$.

While this graphical demonstration is a unique way of finding all of the hyperbolic functions, and also *gd u,* it is not a good way to obtain their values. The reason being that for each choice of the point *P(x,y)* one is required to complete a new construction of the figure. As there are an infinite number of choices for point *P* this graphical approach requiring multiple constructions would be not be practical to use for finding these values.

Two classic applications involving hyperbolic functions

There are literally hundreds of applications of hyperbolic functions found throughout all scientific disciplines. Many of these, I am sure, are familiar to the reader. Over the years I have collected many interesting examples of these applications. Following are two classic examples that are my favorites. They are included here for historical value as one may have heard about them but rarely has had a chance to see them developed.

The first example is taken from Transmission Line Theory. In the late 1800's it was the urgent need for sending electricity over long transmission lines that heralded the modern use of hyperbolic functions. The second example is found in Einstein's, *Special Theory of Relativity* regarding the discussion of the transformation equations relating to different frames of reference. These two are excellent illustrations of the application of hyperbolic functions.

First to be discussed is *Long Transmission Line Theory*. Contrary to what might be supposed, it did not develop because these lines form a Catenary curve as they hang between poles. (You will recall the formula for the Catenary, where $y = a \cosh x/a$, is the hyperbolic solution). Instead, it arose from the second order differential equations that describe the voltage and the current that flows in these lines. The following example showing important steps in the development of the theory is taken from pages 94-101 of the book *Power System Analysis, by* W. D. Stevenson, Jr. (McGraw-Hill, 1955).

The second order differential equations for the voltage (V) and the current (I) are: $dV^2/dx^2 = \gamma^2 \cdot V$, and $dI^2/dx^2 = \gamma^2 \cdot I$; where γ is called the *propagation constant*. (You will note these formulas are of the same general form as the differential equation discussed earlier in this Article). The solution of these by integrating twice is:

$V = (A_1 \cdot e^{\gamma x} + A_2 \cdot e^{-\gamma x})$, and $I = (1/Z_c) \cdot (A_1 \cdot e^{\gamma x} + A_2 \cdot e^{-\gamma x})$; where A_1, A_2, and Z_c (called the *characteristic impedance*) are constants. By using the boundary conditions at the receiving end of the line A_1 and A_2 may be evaluated as follows:

$A_1 = (V_R + I_R \cdot Z_c) / 2$, and $A_2 = (V_R - I_R \cdot Z_c) / 2$.

If we substitute the equations for A_1 and A_2 into those for V and I we obtain:

$V = \{[(V_R + I_R \cdot Z_c) \cdot e^{\gamma x} / 2] + [(V_R - I_R \cdot Z_c) \cdot e^{-\gamma x} / 2]\}$,

And, $I = \{[(V_R/Z_c + I_R \cdot) \cdot e^{\gamma x} / 2] - [(V_R/Z_c - I_R) \cdot e^{-\gamma x} / 2]\}$,

By rearranging terms in V and I more convenient forms of the equations for computing the voltage and current of a power line are found by using hyperbolic functions. We then have: $V = (V_R \cdot \cosh \gamma x) + (I_R \cdot Z_c \cdot \sinh \gamma x)$, and $I = (I_R \cdot \cosh \gamma x) + (V_R/Z_c \cdot \sinh \gamma x)$
Now, these last equations give the voltage and current anywhere along the line. If we let $x = l$ we can obtain the voltage and current at the sending end, as follows

$$V_S = (V_R \cdot \cosh \gamma l) + (I_R \cdot Z_c \cdot \sinh \gamma l), \quad \text{and } I_S = (I_R \cdot \cosh \gamma l) + (V_R/Z_c \cdot \sinh \gamma l)$$

Our work is now finished as these last expressions for V_S and I_S are the fundamental equations of a transmission line. However, they are not as simple as they look as the expression γl is usually complex. This also makes the hyperbolic functions complex, and solutions cannot easily be found using published tables. However, our slide rule with hyperbolic functions can be used to quickly find answers to them.

The second example to be discussed arises from Einstein's, *Special Theory of Relativity*. The hyperbolic functions figure prominently in his theory. They are found in the transformation equations relating to different frames of reference. (This example is taken from pages 81-84 of the book, *Used Math, by* C. E. Swartz (Prentice-Hall, 1973).

The transformation equations relating to x and t to x' and $t'$are

$$x' = (x - vt) / (\sqrt{1} - (v^2/c^2)) \quad and \quad t' = (t - (xv/c^2)) / (\sqrt{1} - (v^2/c^2))$$

The coordinates x' and t' are for a reference frame moving with a velocity v with respect to the frame specifying x and t. There is an invariant interval that has the same value for all frames moving at constant velocity (in the x direction) with respect to each other, so:

$$x'^2 - c^2t'^2 = x^2 - c^2t^2$$

For a particular value of this interval, the relationship between x and t in any frame is hyperbolic. For instance, suppose that we choose a moment in the (x', t') frame when $t'= 0$ and $x'= 1$. Then in the (x, t) frame the relationship between x and t must be such that $1= x^2 - c^2t^2$. The values for x and t lie along a hyperbola where $x = \cosh u$ and $ct = \sinh u$.

These hyperbolic curves are best seen in a graphical form called the Minkowski diagram and first used by him in 1908. It was Minkowski who first proposed the notion of four-dimensional *space-time*. The concept was soon adopted by Einstein and later used by him to develop his crowning achievement the *general theory of relativity*. Thanks to Minkowski, the relationships among the hyperbolic functions and the use of his diagram provide an easy derivation of the relativistic formula for addition of velocities. Here a point on the hyperbola can be given in terms of the parameter u, where

$$x = \cosh u, \text{ and } ct = \sinh u; \quad v = dx/dt = \sinh u \, du/dt, \quad and \; dt/du = \cosh u / c.$$

Therefore, the velocity of the moving frame is $v = c \tanh u$. From within the prime frame of the velocity the double primed frame is $v' = c \tanh u$. The velocity of the double primed frame as seen in the stationary frame is not $v + v'$, but is given by

$$v'' = c \tanh (u + w) = c \sinh (u + w) / \cosh (u + w)$$
$$= c (\sinh u \cosh w + \cosh u \sinh w) / (\cosh u \cosh w + \sinh u \sinh w)$$

If we divide numerator and denominator by $\cosh u \cosh w$ we obtain

$$v'' = c(\tanh u + \tanh w) / (1 + \tanh u \tanh w)$$

so, $v'' = (v + v') / ((1 + (v v'/ c^2),$ and this is the addition law for velocities.

We salute Einstein, and his *Special Theory of Relativity,* and Minkowski for a great display of the use of hyperbolic functions. Again, instead of using tables, one would find that the answers these equations could best be solved using our hyperbolic slide rule.

History of Slide Rules with Log Log and Hyperbolic Function Scales

In the early decades of the 1900's, as the expansion of hyperbolic applications accelerated, it seemed only a matter of time until the slide rule emerged as an aid for the Electrical Engineer. In JOS Vol. 1, No. 1, there is a paper by Bob Otnes on the *Log Log Scales* (The comments that follow are based on information taken from his paper).

In 1815, Peter M. Roget, MD, invented the log log Scales. Then, almost 100 years were to elapse before log log slide rules would come into regular use. About 1906, both John Davis & Son and A.W. Faber introduced slide rules with log log scales, and K&E followed in 1909. The importance of log log (LL) scales is that they are useful in evaluating exponential expressions such as x^y (i.e. like $7.1^{2.3}$ and $e^{1.83}$). Being able to evaluate the powers of $e^{\pm y}$ allows the calculation of hyperbolic functions. So slide rules with LL scales were the forerunners of the slide rules that came later with scales of hyperbolic functions. Before slide rules with LL scales were introduced the only practical means to evaluate exponential functions were by using published books with log tables. Or, by calculating $e^{\pm y}$ from the ordinary L scale. Now, having for use only the log tables or the L scale can often involve tedious calculations with the possibility of errors. So these early slide rules with their LL scales saved a lot of time and labor. Equally important is that they provided a ready means to check the calculations that had been made by hand from the published log tables.

The steps of the mathematical theory behind the LL scales are this:

Start with the expression x^y. Its log is, $\log x^y = y \log x$

Then the log of this is, $\log (y \log x) = \log y + \log (\log x)$.

So we have a resulting expression that becomes one of addition when these log log steps are calculated. Now, this equation can be used to apply to any logarithmic base "*a*" that we choose. Roget used base $a = 10$. John Davis & Son used $a = 2.0$, and A.W. Faber used $a = 3.08$. In 1909 K&E introduced their Log Log slide rule with $a = e = 2.71828$….. Many problems in science and engineering involve *e.* As a result of K&E's development of the LL scales with *e* as a base, their design became the industry standard - and this base was used on most LL slide rules that came later. An excellent article on the *"Theory Underlying Construction of the " LL" Scales*" is found in the Dietzgen Manual for Model No. 1732, the Decimal Trig Type Log Log slide rule (pages 92-94).

One of the earliest examples I have seen of the calculation of hyperbolic functions using the LL *scales* is the solution of the Catenary curve on page 6 of the K&E 4092 manual (© 1914). The table on that page shows the calculations for the Catenary using nine different examples of $e^{\pm x/a}$. We see clearly from this table that one does not need a slide rule with hyperbolic scales to obtain the values for these functions. In fact, many well known slide rules that do not have hyperbolic scales have a section in their manuals on how to obtain hyperbolic function values. Manual examples of this are the VERSALOG (pages 29-30), and the DECI-LON (pages 107-108) - and all of the Log Log Trig manuals that I have in my files from the various manufacturers (K&E, Dietzgen, Pickett, and Post) have sections on obtaining the powers of "*e*". However, it is somewhat inconvenient using the LL scales on these rules for calculating the hyperbolic functions, as to do this takes four separate steps. For example, to calculate Cosh x/a one has to first obtain the values for $e^{+x/a}$ and $e^{-x/a}$, then add them together and divide the result by 2. For Sinh x/a you subtract $e^{-x/a}$ from $e^{+x/a}$, and divide the result by 2. Complicating this process is the fact that to keep track of the calculations you usually have to record them by pencil and paper. These multiple steps using the LL *scales* are just not as convenient as having a slide rule with hyperbolic scales to use for the direct reading of the hyperbolic function values – and to use for calculations involving these values.

I am indebted to Rodger Shepherd, for information on the early history of slide rules with scales that allowed the reading of hyperbolic function values. In JOS Vol.4, No. 2, he gave a description of an early design by F. Blanc in 1890. It did not have hyperbolic scales, but it did have marks on it so when the conversion to base "e" was made one could read off the Sinh and Cosh values. Unfortunately no picture of this slide rule exists. Rodger also sent me a copy of a February 15, 1921 paper, by J. St.Vincent Pletts, that was published in the *Proceedings of The Physical Society of London*. Its title was *Some Slide Rule Improvements*, and seems to be the first to show an actual picture of a slide rule with hyperbolic scales.

Next we find the submission of a Patent application on May 12, 1921 by Albert F. Puchstein. He was a professor at Ohio State University. The title was *"Device For Making Vector Calculations",* and included layouts of hyperbolic scales. In the patent application Puchstein says; *"…….my device is of such a nature that calculations can be readily made as to ….. hyperbolic sines, cosines, tangents, etcetera, of vectors".* The patent was approved three years later on March 25, 1924 (U.S. No. 1,487,805). Shortly after this Puchstein discussed his slide rule design with K&E, but nothing happened as K&E felt then that there was no market for such a rule.

There may have been other articles published in these early years by other authors, but if written they seem to be lost. So, we find that none of these early designs of hyperbolic slide rules were ever manufactured or sold to the public.

The history of slide rules with hyperbolic scales that were actually sold to the public begins with a copyrighted paper published in May 1928 by Mendell P. Weinbach. He was a Professor at the University of Missouri-Columbia. This was titled, *"Vector Calculating Devices"* (A.I.E.E. Journal, V.47, May 1928, pages 336-40). In this paper he showed a picture of *"The Vector Slide Rule".* Professor Weinbach was mainly responsible for the design of the K&E 4093-3, Log Log Vector slide rule, with hyperbolic scales. He also was the sole contributor to the drafting of its manual. The K&E 4093-3 was introduced in late 1929. It was first listed for sale in the 1930 K&E catalog. The rule's price was $16.00. The 4093-3s version, with a better leather case, was $16.85.

Ten years later, in 1939, K&E introduced a second Vector slide rule; the K&E 4083-3 Log Log Duplex Vector slide rule. This was an improvement over the earlier 4093-3 model. Again, Professor Weinbach was solely responsible for drafting the new manual. A complete history of *"Mendell Penco Weinbach and the K&E Log Log Vector Slide Rules"*, the K&E 4093 and K&E 4083, may be found in my March 2008 Article in the Oughtred Society's JOS Plus web site: http://www.oughtred.org/josplus.shtml

Referring back to 1929 and the K&E 4093-3, the next hyperbolic slide rule introduced after it was the Hemmi Model 153. This was introduced in 1933 in Japan. The Patent Application was submitted by Hasashi Okura in Japan on January 14, 1932, and in the U.S. on March 20, 1933 (The U.S. Patent No. 2,079,464 was granted on May 4, 1937). This rule did not have hyperbolic scales but instead had *Gudermannian* scales for obtaining Hyperbolic Functions (spelled Gudermanian by Hemmi). Its 1934 manual describes this model as the *Electrical Engineer's Universal Duplex Slide Rule With*

Patent Vector Scale, Gudermanian Scale and Log Log Scales. This is a very usable rule and is one of the most uniquely designed of all the hyperbolic slide rules. (As an interesting side note, US Patent No. 2,086,502 was approved later in 1937. This was issued to Okura on behalf of Hemmi and K&E. It was for *"Gudermannian Scales for Hyperbolic Functions"*. Although the patent was approved in 1937, K&E never introduced a slide rule with *"Gudermannian Scales"*).

After the introduction in 1933 of the Hemmi Model 153 about another fifteen years passed before the Dietzgen Model 1735 and the Pickett & Eckel Model 4 arrived on the scene. Their manuals are both dated with a 1948 Copyright. The European makers were much slower in introducing slide rules with hyperbolic scales. The following dates are estimates of the earliest dates their slide rules were introduced:

- The Aristo HyperboLog 971 appears to be the first marketed in Europe in 1954.
- The Blundell JV56 Multi-Log Vector Duplex was second in 1957.
- About 1962, the Graphoplex 691a Neperlog Hyperbolic, appeared.
- The Faber-Castell 2/84 Mathema made its debut in 1966.
- The date for the Reiss 3227, made in East Germany, also appears to be 1966.
- The larger Aristo, the HyperLog 0972, followed in 1969.

These slide rules were known as "high-end ones", as when introduced they usually were the most expensive and had the most scales of slide rules in a Company's product line.

Of interest is the fact that the first appearance (1954) of the hyperbolic slide rule in Europe was 25 years after its introduction by K&E in the USA. During this period only limited shipments of slide rules occurred from the European and Japanese manufacturers to the U.S.A. Much of this delay was due to international tensions prior to World War II. This was then followed by more years during WWII, and after the War with the rebuilding of Europe and Japan. As a result, K&E had a virtual monopoly on hyperbolic slide rule marketing in the United States for about 19 years. This continued from 1929 until around 1948 when Dietzgen and Pickett introduced their hyperbolic slide rules.

When we look at the world-wide sales of all types of slide rules it does not appear that there was ever any real penetration by K&E, or other U.S.A. slide rule makers, into the European or Japanese markets. The same can be said about the European manufacturers regarding the U.S.A. market. Hemmi did have some limited success selling its own models in the U.S.A., before and after WW II; and it did fairly well after the War with its arrangement with Post in the 1950's. Also, some European cross production of U.S.A slide rule names occurred at various times. But for all practical purposes, this was never on a significant scale. The U.S.A. manufacturers, K&E, Dietzgen, and Pickett, remained dominant throughout all of the years in the U.S.A. market. Hemmi in Japan, and the European manufacturers, Dennert & Pape, Nestler, Faber-Castell, Blundel, and others remained dominant in Europe.

To my knowledge, neither Dennert & Pape, nor Nestler ever manufactured a slide rule with hyperbolic scales under their well known names. Although D&P started using the product name Aristo in 1936, it was not until 1954 that the Aristo HyperboLog 971 was marketed. Also, the Faber-Castell Mathema 2/84 did not come on the scene until about 1966. I have no explanation as to why the European makers were so late introducing slide rules with hyperbolic scales. This is a similar puzzle as to why the U.S. Manufacturers never adopted the P scale that had been introduced in 1935 by F-C into the System Darmstadt configuration. I had thought the reasons for these omissions might be due to patent infringement concerns. However, John Mosand, indicated to me that patent worries were not a problem. John says, “Clearly, there must have been conservative attitudes on both sides of the Atlantic”. Maybe some of you Readers could offer other reasons for these omissions, both for Europe and the U.S.A.

In the later years of slide rule production we find many names of hyperbolic Chinese slide rules on the list. In spite of what seems to be a large number of these that were manufactured it appears few ever found their way to Europe or the U.S.A.

The history of hyperbolic slide rules in the U.S.S.R. is surprising. A few years ago Andrew Davie referred me to a friend of his who was an expert in Russian slide rules. His name is Sergei Frolov, an engineer and computer expert who had a web site showing Russian slide rules. Sergei informed me that he had never seen a Russian slide rule with hyperbolic scales. This appears to be the case, as to date I have not seen one either.

Some Practical Pointers About Slide Rules With Hyperbolic Scales.

The length of the scales on most of the rules is 25 cm or 10 inches. I found in actually measuring some of them that the manufacturers were not always correct in reporting the length of the scales. So in my listing I left the lengths as described rather than making changes. Both Hemmi and K&E produced larger 20 inch (50 cm) versions of some of their hyperbolic slide rules. I found only two pocket versions: the Flying Fish 1200, and the Pickett N4p in identical T and ES versions. One of the rarest is the Eckel Engineer’s Log Log Circular slide rule (size about 8 inches). It is the only circular slide rule known with hyperbolic scales. This was made by Eckel after he left Pickett.

Most of the rules have designated hyperbolic Sinh and Tanh scales. (There are only a few exceptions. These include the Hemmi 153, and the similar rules from other makers, that have *Gudermannian* scales for obtaining hyperbolic functions). Usually the Sinh and Tanh scales are shown as Sh1, Sh2 and Th. Almost always there are two hyperbolic Sh scales to give a wider range for accuracy. Only a few slide rules have Cosh scales.

The ranges of the Hyperbolic Scales usually are: Sh1 from 0.1 to 0.882; Sh2 from 0.882 to 3.0; and Th from 0.1 to 3.0. If a Cosh scale is included on the rule the range of values is usually Ch from 0.1 to 3.0. For those rules that do not have a Ch scale its value is calculated by use of the formula Ch = Sh / Th.

The hyperbolic scales are found in different positions on the slide rules. On some all of the scales are positioned on the top or on the bottom sections of the stock. On some rules the scales are split between the top and bottom stock sections. One notable exception is the Pickett & Eckel Model 4 rule with its hyperbolic scales on the slide and not the body.

Not all hyperbolic slide rules are created equal. We find the layout of the scales on the various slide rules can be important for ease of operation. Also, accuracy can suffer if multiple alignments of the scales and the cursor hairline are involved in the steps when calculations are made. It is obvious that accuracy could be lost if you have to turn the rule over to read a result, or rely on the hairline on the cursor to read from the top scales to the bottom scales on the stock. To minimize concern about alignment of the hairline with the scales an ideal slide rule design would have the D, DI, Sh1, Sh2 and Th scales together on the bottom of the stock, with these adjacent to C and CI scales on the slide. The CI and DI scales are required for calculating the reciprocal hyperbolic functions. The need for this layout is also obvious when Ch is calculated. The reason for this is that to obtain

Ch you first have to find the Sh and Th values, and then calculate Ch = Sh/Th. Of course this step is not needed if a Ch scale is included. However, as only a small number of the rules on the list have a Ch scale, most of the time the Ch value will have to be calculated using the Sh and Th amounts.

As I worked on my listing showing the details of the scales and gauge marks on the various hyperbolic slide rules it was a surprise to find that most were poorly designed. This was where the hyperbolic scales were not placed close to a set of C/D, and CI or DI scales. Also, poor layouts were found where the Sh and Th scales were separated from top to bottom. This meant that the cursor hairline and the top and bottom scales had to be perfectly aligned in order to read the most accurate values.

There are many poor designs by the various Manufacturers. Examples are the K&E 4093-3, and an early version of the K&E 4083-3. These have no C scale on the same side as the hyperbolic scales. To read the Ch value, after getting the Th and Sh values, the manual instructions say to turn the rule over and read the answer at the cursor hairline on the C scale. K&E should have given other instructions for these rules for there is a much better alternative way to obtain a more accurate Ch value. All you need to do, before starting work to find any of the hyperbolic functions, is to remove the slide, turn it over and reinsert it. This places the C scale on the opposite side next to the D scale. Also, you obtain the added advantage of the use of the CI scale for the reading of the Csch, Sech, and Coth values. The accuracy for these slide rules (and others with similar scale layouts) is greatly improved with this little trick. With the slide reversed on these rules the layout becomes almost ideal, and the ability to handle more accurate calculations involving hyperbolic functions is greatly enhanced.

Two slide rules are almost at a tie for ease of use in finding the Ch value. These are the Hemmi No. 153 and the Pickett & Eckel Model 4. The Hemmi has *gudermannian* scales in its design. With the slide reversed on this slide rule you can read off the Sh and Th values on the T and Q scales with one setting on the G_θ scale. Then sliding the hairline to the Sh value on the Q scale you can read the Ch value on the Q' scale. The Pickett & Eckel Model 4 has the hyperbolic scales on the slide. You do not have to reverse the slide on this rule to find Ch. You simply align the C and D scales. Then set the cursor line on the Sinh scale value and slide the Tanh scale under the cursor line. The answer for Ch will be on the D scale under 1 index on the C scale.

On the other slide rules the manual instructions for calculating Ch, and the reciprocal values may vary depending on the scale layout of the particular rule. Unfortunately, many of the instructions given in these manuals are poor and only sketchy. The best hyperbolic functions instruction manuals I have found are those for the Dietzgen No. 1725 and No. 1735 rules. The instructions are almost identical for these two manuals, and in my research exceed any manual produced by others.

The usual procedure to find a hyperbolic function value is to start by moving the hairline to the value on the appropriate hyperbolic scale, and then read the answer on the C or D or CI scale. We will use the K&E 4083-3 (with the slide turned over and reinserted so that we will have the use of an adjacent C scale) for showing examples. This K&E model is used here because it is found in more numbers than other Company's models. However, the following procedures may be used on almost all makes of rules other than *gudermannian* ones.

For our examples we will use the value u = 0.4 and show how to find all of the hyperbolic functions starting with Sinh 0.4 = ?. With this slide rule you move the hairline to 0.4 on the Sh1 scale and then read 0.411 under the hairline on the D scale. (Note that radians are used, not degrees, for entering the values on the hyperbolic scales). At the same time, if the C and D scales are aligned, you can read Csch 0.4 = 2.43 on the CI scale under the hairline. Similarly, we find Tanh 0.4 to read 0.380 under the hairline on the D scale. At same time you can read Coth 0.4 = 2.63 on the CI scale under the hairline. This slide rule does not have a Cosh scale. To find Cosh 0.4 move the hairline to 0.4 on the Th scale, slide the index 1 on the left side of the C scale to under the hairline, move the hairline to 0.4 on the Sh1 scale, and under the hairline read Cosh 0.4 = 1.08 on the C scale. At the same time you can read Sech 0.4 = 0.925 on the CI scale under the hairline.

Since the ranges of the scales are limited they cannot be used to find answers for all values of u. However, estimates may be used to find approximate results for low values (u < 0.1); and the Log Log scales of the slide rule may be used for obtaining high values (u > 3.0) of the hyperbolic functions. To do this the following estimates may be used:

When *u* < 0.1; *Sinh u = u, and Tanh u = u, and Cosh u* = 1.0.

When *u* > 3.0; *Sinh u =* $(e^u)/2$*, and Tanh u* = 1.0, *and Cosh u = Sinh u.*

Here is an example for Sinh u, where u = 6.0. For this we will use the C and LL3 scales with the slide in its usual position. On scale C put hairline on 6.0. Read 403.0 on the LL3 scale. Then divide by 2. So Sinh 6.0 = 201.5 (The actual value from the tables is Sinh 6.0 = 201.7132, and Cosh 6.0 = 201.7156). Then Tanh 6.0 = (Sinh 6.0/Cosh 6.0) ≈ 1.0.

If you have a slide rule with hyperbolic scales and want to practice, you can the use the following table to check your calculations. It will give you an idea as to where the decimal points are to be placed when doing actual problems (See previous pages for steps to calculate the Cosh on the K&E 4083, Hemmi No.153, and Pickett & Eckel Model 4).

Short Table of Hyperbolic Functions

u	Sinh u	Cosh u	Tanh u	Csch u	Sech u	Coth u
0.2	0.2013	1.0201	0.1974	4.9668	0.9803	5.0665
0.4	0.4108	1.0811	0.3799	2.4346	0.9250	2.6319
0.8	0.8881	1.3374	0.6640	1.1260	0.7477	1.5059
1.4	1.9043	2.1509	0.8854	0.5251	0.4649	1.1295
2.4	5.4662	5.5569	0.9837	0.1829	0.1800	1.0166
2.8	8.1919	8.2527	0.9926	0.1221	0.1212	1.0074
3.0	10.0179	10.0677	0.9951	0.0998	0.0993	1.0050
6.0	201.713	201.716	1.0000	0.0050	0.0050	1.0000

The Slide Rule and Hyperbolic Functions of Complex Numbers

In this section we will show the significant impact the introduction of the K&E 4093 slide rule with hyperbolic scales had on problems involving complex numbers. Weinbach's design that K&E used was an important milestone in slide rule history. This was for the reason that Weinbach was able with his rule to combine the movement of the hyperbolic scales with that of the decimally divided trig scales in order to solve complex hyperbolic functions. With his rule, values of complex hyperbolic expressions such as

$sinh\ (u + j\ \theta) = A\ /\underline{\alpha}$, in the *Polar* form ; or, $sinh\ (u + j\ \theta) = (x + j\ y)$, in the *Cartesian* form could now be quickly obtained. Compared with the then current hand calculation routine by logarithms this breakthrough was a most valuable and timesaving tool as we will see in the following discussion. But first, without getting too deep in the mathematics, we will give a short explanation of what Weinbach was facing in designing a slide rule to solve these problems.

A typical complex hyperbolic number problem would involve either the calculation of $A\ /\underline{\alpha}$ in the *Polar* form, or, the calculation of the values of x and y in the *Cartesian* form. For example, the formulas for solving $sinh\ (u + j\ \theta) = A\ /\underline{\alpha}$, in the *Polar* notation, are;

$$A = \sqrt{(sinh^2 u + sin^2 \theta)},\ or\ A = (sinh\ u \cdot cos\ \theta / cos\ \alpha);\ and$$

$$\alpha = tan^{-1} (cosh\ u \cdot sin\ \theta / sinh\ u \cdot cos\ \theta);\ or\ \ \alpha = tan^{-1} (tan\ \theta / tanh\ u).$$

If instead one wished to solve for the *Cartesian* values of x and y the following formulas would be used:

$$sinh\ (u + j\ \theta) = (x + j\ y) = [(sinh\ u \cdot cos\ \theta) + j\ (cosh\ u \cdot sin\ \theta)].$$

Similar looking *Polar and Cartesian* formulas exist for solving the other complex hyperbolic functions, i.e., $cosh\ (u + j\ \theta)$, $tanh\ (u + j\theta)$, and the reciprocal functions $cosech\ (u + j\ \theta)$, $sech\ (u + j\ \theta)$, *and* $coth\ (u + j\ \theta)$.

Now the challenge Weinbach had was to design a slide rule that would handle all of these various and complicated calculations – including those for all six of the complex hyperbolic functions. Notice that the solution shown for each equation above involves one, or more, of the trigonometric functions combined with hyperbolic functions - and, might include other arithmetic operations as well. This means there must always be trigonometric scales working along with the hyperbolic scales on the slide rule in order to solve these complex hyperbolic expressions. Weinbach was able to devise his slide rule so that these different sets of scales would provide the needed solutions. This was a significant achievement.

In the days before Weinbach's slide rule these formulas involved formidable calculations. To solve these equations one had to consult published mathematical tables. Then look up the hyperbolic functions in one set of tables and the circular functions in another. (Remember, the hyperbolic functions were looked in the tables in radians, but the circular functions had to be in degrees before table look up. One or the other, *u or* θ, had to be converted to degrees or radians before using the tables). Complicating the calculations was the fact that these look ups were almost always required to be in log tables – and often interpolations within the tables were encountered that added to the difficulty of the work. Log tables were used for these calculations because of the multiplication and division operations in the formulas. Anyone who has worked with logs and anti-logs knows they often are confusing to use. Calculations are time consuming and one has to very careful not to make mistakes. However, there was little alternative at the time, for without log tables to use the calculations would have had to be done by hand using long multiplication and long division.

After 1914 the work may, or may not, have been simplified if one had a copy of the "Tables of Complex Hyperbolic and Circular Functions", by Kennelly. Using his tables one could obtain approximations to the values - and more importantly a means to check the results obtained by the long hand calculation process. These were a good tool for the engineer to have, but did have limitations. Alfred Still in his book, "*Electric Power Transmission*", on page 290, said this about the Kennelly tables; "Although tables of complex hyperbolic functions are available, they should preferably be used only by those who have a thorough understanding of their mathematical basis and are familiar with their use. Moreover, since an enormous number of values would have to be included in order to cover all possible combinations of the components u and v in the complex expression ($u + j\,v$), it is usually necessary to resort to interpolation which is not only tedious, but likely to introduce errors when attempted by those who have not thoroughly familiarized themselves with the methods by constant use of the tables".

In his 1928 article Weinbach agreed with this by saying, "The 'Kennelly Tables' give the vector values and the equivalent complex numbers of the above mentioned functions for values of u in steps of 0.05 and values of θ in steps of 4.5 deg. Double interpolations are necessary, however, if the values of u and θ differ from those given in the table". From these two comments we see that most of the time the table look ups would involve double interpolations that were not easy to do.

As we will see, Weinbach's introduction of his slide rule with hyperbolic scales presented a powerful tool for both making and checking these complex hyperbolic calculations, and in much less time. To show the power of his slide rule we will first show a calculation of $\sinh (u + j\,\theta) = A\,/\underline{\alpha}$ by the hand calculation method. Then four different slide rule examples will be shown comparing the solution by the long hand calculation method with the solutions by slide rule.

There is little doubt, as we will see, that the introduction by K&E in 1929 of Weinbach's slide rule with hyperbolic scales was a major breakthrough in slide rule history. For nineteen years, from 1929 to 1948, K&E had a monopoly as no other manufacturer had a similar slide rule with hyperbolic scales in the market. During that time the two K&E slide rules became the leading tool for engineers and scientists to solve and check calculations involving both hyperbolic and complex hyperbolic functions. It should be mentioned that Hemmi from the early 1930's had two hyperbolic rules. Model No. 153 of Gudermannian design, and the 20 inch Model No. 154. However, they really were not competitive with the K&E rules as one could not directly or easily solve the many problems involving complex hyperbolic functions with them. (The term "directly" means solving problems in one continuous set of operations of the slide rule, and not having to stop to record a result, or to reset the rule to a different value, between steps).

After World War II Hemmi introduced their Model No. 255 Duplex Slide Rule. With this slide rule one could solve complex hyperbolic functions directly. This was the first slide rule that could match the operations of the K&E 4083. However, it and the other Hemmi rules only appeared in the U.S. market in very limited numbers so they were never a challenge to K&E's market dominance. In fact, it was not until 1948 that Pickett & Eckel, and Dietzgen began to introduce their models with hyperbolic scales in sufficient numbers to compete with the K&E 4083 slide rule. Of interest is that the Pickett & Eckel Model 4 had the hyperbolic scales on the slide similar to Weinbach's original design in his 1928 paper. The Dietzgen No. 1735 looked so like the K&E 4083 that many mistakenly thought K&E had manufactured it.

In making our comparisons we will start with an example using the hand calculation method, and then move on to the calculations by slide rule. For the first slide rule example we will use the original K&E 4093-3 (1929); for the second we will use the Hemmi 153 (1933); for the third we will use the K&E 4083-3 slide rule (1939); and, for the fourth we will use the Pickett Model 4 (1948). I did not intend to limit the examples to only these four rules. However, these seemed to be enough different designs for us to see the various methods of the slide rule calculations.

In our examples we will solve ***sinh (u + j θ)* = A /α**, using sample values for ***u*** and ***θ***, and these formulas:

$$\alpha = \tan^{-1}(\tan\theta / \tanh u); \text{ and, } A = (\sinh u \cdot \cos\theta / \cos\alpha).$$

We will pretend that the time period in which we are doing the hand calculation is pre-1930, before the introduction of Weinbach's slide rule. Our calculation will be done using published log tables that were usually on every engineer's desk at that time. At least on the desks of those doing these types of calculations. For our example we will use the following tables to look up the values by hand: For the regular trigonometric (circular) functions we will use the *Handbook of Chemistry and Physics*, 1929; for the logs of the trigonometric functions we will use *Logarithmic Tables of Numbers and Trigonometrical Functions*, by Vega, 1856; and, for the logs of the hyperbolic functions we will use *Hyperbolic Functions* (*Smithsonian Mathematical Tables),* by Becker and Van Orstrand, 1909).

Following is our hand example, showing the look up sources used, and the steps involved in the calculation of ***sinh (u + j θ) = A /α***, where **u = 0.243** and **θ = 53° 30'**:

For α =	***tan⁻¹ (tan θ / tanh u)***	**Look up Source**	**Log Value**	
Step 1.	*log tan θ =*	Vega p 509	0.1307911	
Step 2.	*log tanh u =*	B & VO p 24	9.3771700	-10
Step 3.	*log tan θ = ± 10*	Step 1 ± *10*	10.1307911	-10
Step 4.	*log tan θ - log tanh u*	*= Step 3 – Step 2*	0.7536211	0
Step 5.	*anti-log (tan θ / tanh u) =*	Vega p 99	5.67045	
Step 6a.	***α =*** *tan⁻¹(Step 5)*	HB of C&P- P103	**79° 59.9'**	
Step 6b.	***α =*** (from Step 6a. to decimal °)		**79.99833**	°

For A =	***(sinh u · cos θ) / cos α***			
Step 7.	*log sinh u =*	B & VO p 24	9.3898700	-10
Step 8.	*log cos θ =*	Vega p 509	9.7743876	-10
Step 9.	*log sinh u + log cos θ*	*= Step 7 + Step 8*	19.1642576	-20
Step 10.	*- 10*	Step 9 - *10*	9.1642576	-10
Step 11.	*log cos* **α** = (**α** is from Step 6)	Vega p 350	9.2397000	-10
Step 12.	*log A = log sinh u + log cos θ - log cos α*	= Steps 10 – 11	-0.0754424	0
Step 13.	*log A* = ± 10 *(* from Step 12)		9.9245576	-10
Step 14.	***A =*** (anti-log of *log A* from Step 13)	Vega p 154	**0.84054**	

So, the solution of ***sinh(u + j θ)* =A/α** is; **sinh (0.243 + j 53° 30′) = 0.84054 / 79.99833°**

If you actually do these steps by looking up the values in the tables you will very quickly see that this hand calculation method is not simple or easy. It is really quite laborious. Remember that back then all calculations were recorded by hand using pencil and paper. Starting with the books of tables in front of me, and a pencil and pad of paper, it took 27 minutes for me to complete this hand calculation – and recheck the work. I would be interested to hear from readers as to how long it took them to complete this same example using published tables.

Slide Rule Example No. **1** using the Log Log Vector, K&E 4093:

Below is the example using the K&E 4093 slide rule to solve *sinh (0.243 + j 53° 30′)* for *A /*α. First we will convert 53° 30′ to decimal degrees of 53.5º. (Note: to save space we will use the symbol ↨ to denote the use of the hairline on the cursor).

For the direct solution of α the steps are :

(1). With the scales aligned set ↨ to 53.5º on scale *TI* black;

(2). Move right index on scale *Sl1* to 0.243 on the *Th* scale;

(3). Under ↨ on scale *TI* black read 80º. So, **α = 80º**.

For the direct solution of **A** the steps are :

(4). With the scales aligned set ↨ to 0.243 on *Sh1*;

(5). Move slide so that 53.5º on scale *Sl2* black is under the ↨;

(6). Move ↨ to 80º on scale *Sl2*;

(7). Read value of 0.84 under ↨ on scale *D; So,* ***A = 0.84***

And, the solution by the K&E 4093 slide rule is:

sinh (0.243 + j 53.5º) = 0.84 / 80º

Starting with the K&E Log Log Vector instruction manual to guide me, and a pencil and pad of paper, it took 4 1/2 minutes for me to complete this slide rule calculation.

Slide Rule Example **2** using the Hemmi 153:

For the second example using the slide rule we will use the Hemmi 153 introduced in 1933. Although you cannot read results directly as with the K&E 4093-3 we can still solve *sinh (0.243 + j 53° 30′)* for A /α. We do this by first obtaining the values for *cos θ, tan θ, sinh u, and tanh u.* Then we will use these separate values to solve the equations:

$$\boldsymbol{\alpha} = \tan^{-1} (\tan\theta / \tanh u); \text{ and, } \boldsymbol{A} = (\sinh u \cdot \cos\theta / \cos\alpha)$$

For α the steps are :

(1) for tan 53.5º ; set ↨ to 53.5 on *θ*, and read 1.35 on *T* other side under ↨;

(2) for tanh 0.243; set ↨ to 0.243 on G_θ, and read 0.24 on *P* other side under ↨;

(3) for *α* ; set ↨ to 1.35 on D and move 0.24 on C under ↨, under right index of C read 5.63 on D;

(4) for *α* ; then set ↨ to 5.63 on T, read 80.0º on *θ* other side under ↨. So, **α = 80º**

For **A** the steps are :

(7) for sinh 0.243; set ↨ to 0.243 on G_θ, and read 0.245 on *T* under ↨;

(8) for cos θ 53.5º; set ↨ to 36.5 (90.0 – 53.5) on *θ*, and read 0.594 on *P* under ↨;

(9) for cos 80.0º*;* set ↨ to 10.0 (90.0 – 80.0) on *θ*, and read 0.174 on *P* under ↨;

(10) for A; set ↨ to 0.245 on D and move 0.174 on C under ↨, move ↨ to 0.594 on C and read 0.84 on D under ↨; So, **A = 0.84**

And, the solution by the Hemmi 153 slide rule is: **sinh (0.243 + j 53.5º) = 0.84 / 80º** .

It took 6 minutes for me to complete this slide rule calculation.

Slide Rule Example No. **3** using the K&E 4083:

We will now show the solution steps using the K&E 4083 slide rule, introduced in 1939, to solve *sinh (0.243 + j 53° 30′)* for A /α.

For the direct solution of α the steps are :

(1). set ↨ to right index on *D* and move 53.5º on *T (red)* under the ↨;

(2). move ↨ to 0.243 on *Th*, and under ↨ read 80.0º on scale *T (red)*; so, **α = 80º**

For the direct solution of **A** the steps are :

(3). set ↨ to 0.243 on *Sh1*;

(4). set 80.0º on scale *S (red)* under the ↨;

(5). move ↨ to 53.5º on scale *S (red);*

(6). read value of 0.84 under ↨ on *D.* So, ***A = 0.84***

And, the solution by the K&E 4083-3 slide rule is: **sinh (0.243 + j 53.5º) = 0.84 / 80º**

It took 3 minutes for me to complete this slide rule calculation. Notice how much easier and faster the steps have become using the K&E 4083 slide rule vs. those used for the K&E 4093.

We solved for the complex Vector form *A /α* when doing the above example. This form is more often preferred as it is particularly useful for multiplying and dividing these complex hyperbolic numbers. As another example we will now show a solution for x and y using the Cartesian form. Here we will solve *sinh (0.243 + j 53.5º) = (x + j· y)*. The formulas for solving for x and y are: $x = (\sinh u \cdot \cos \theta)$; $y = (\cosh u \cdot \sin \theta)$;

The solution of these equations for x and y by the long hand method using logarithms is: x = 0.14597 and y = 0.82771. To save space we will not show the steps here of the detail listing of these hand calculations using logarithms. I did do them by looking up the logs in the published tables and it took me 18 minutes to solve for these by hand using pencil and paper. Using the K&E 4083-3 slide rule we will proceed as follows:

For the direct solution of **x** the steps are :

(1). set ↨ to 0.243 on *Sh1*;

(2). set right index on *S* to ↨ ;

(3). move ↨ to 53.5º on scale *S (red);*

(4). read ***x = 0.146*** under ↨ on *D*

For the direct solution of **y** the steps are :

(5). set left index of *C* to 0.243 on *Th*;

(6). move ↨ to 0.243 on *Sh1*;

(7). turn rule over read 1.03 on *C* scale under ↨**;**

(8). turn rule back over and set left index of *S* to 1.03 on *D*;

(9). move ↨ to 53.5 on *S black*;

(10). read ***y = 0.828*** under ↨on *D*

And so, **sinh (0.243 + j 53.5º) = (0.146 + j · 0.828)**

These slide rule calculations for x and y took 4 minutes for me to complete.

Slide Rule Example **4** using the Pickett & Eckel Model 4:

For our fourth example, to solve *sinh (0.243 + j 53° 30')* for A /α , we will use the Pickett & Eckel Model 4 slide rule introduced in 1948. This slide rule is a different design as it has the hyperbolic scales on the slide and not the body.

For the direct solution of **α** the steps are **:**

(1). with the C and D scales aligned set ↨ to 53.5º on T;

(2). turn rule over and move slide left until 0.243 on Th is under the ↨ ;

(3). move ↨ to right C index;

(4). turn rule over, align C and D together and read 80.0º on T; so, **α = 80º**

For the direct solution of **A** the steps are :

(5). with the C and D scales aligned set ↨ to 0.243 on Sh

(6). move slide until 80.0º on S (red) is under the ↨;

(7). move ↨ to 53.5º on S (red) and read 0.84 on D; So, **A = 0.84**

And, the solution by the Pickett & Eckel Model 4 slide rule is:

sinh (0.243 + j 53.5º) = 0.84 / <u>80º</u>

These slide rule calculations took 3.5 minutes for me to complete.

It is very obvious from these four examples that using any one of these slide rules is much easier and faster than trying to calculate *sinh* ($u + j\theta$) = $A\ /\underline{\alpha}$, or = ($x + j\ y$), by hand calculations using values from published tables. Of real importance is the fact that it does not take very many sample calculations with the slide rule to find that one masters the solution steps quite readily. One can imagine how happy those working with hyperbolic complex functions were to see these rules. They freed them from many time consuming log table look up routines.

There was another clear advantage. This occurred when more accuracy (more decimal places) was desired than could be obtained by using the slide rule. Then the published tables would have been used, instead of the slide rule. However, after completing the long hand calculations made by using the tables the slide rule could then be used to quickly check the answers. In this way it proved to be a valuable checking tool to have, for often one would find that the lengthy log calculations done by hand were prone to error.

Again, I feel there is the need to say something about the instructions given in the various manufacturers' manuals. They were a real disappointment to me. Unfortunately, I found the instructions for solving complex hyperbolic functions in the manuals of most of the well known slide rules to be difficult and confusing to understand. The best instructions I have found are in the Dietzgen Model 1725 and 1735 manuals (Either manual may be used as they are almost identical). These were much better and easier for me to follow than those in the K&E, Hemmi, Pickett & Eckel, and the European manufacturers' manuals.

Why are Slide Rules with Hyperbolic Function Scales called Vector Slide Rules?

In December 2002, Marion Moon, asked the ISRG members "*what vendors had in mind when they used "vector" to name a slide rule?*" Maybe the following comments will help answer that question.

Mike Konshak has an extensive and wonderful web site by the name of the "International Slide Rule Museum": http://www.sliderulemuseum.com/. He is a well known slide rule authority. On his site is a section titled "Slide Rule Terms, Glossary and Encyclopedia". If one looks up *"Vector Slide Rule"* on this list its definition is: "**Vector Slide Rule** - A slide rule with **hyperbolic** functions **Sh**, **Th**, etc". This seems to be the accepted universal definition as one finds it repeated elsewhere in the slide rule literature.

However, it seems an odd choice of a description as every engineer, mathematician, and scientist who has worked with a slide rule knows you do not need hyperbolic scales to make vector calculations. In fact all you need are the S and T scales along with the C and D scales. Vectors are points in space and can be expressed in the *Polar or Cartesian* coordinate systems. The *Polar* system determines the point by an angle and distance. Whereas the *Cartesian* system determines the point by its x and y coordinates.

However, the use of the term "Vector" for these slide rules does not come from this well known source. Instead, it comes from a different meaning that is commonly used by Electrical Engineers. Many of the important formulas the Electrical Engineer encounters involve vectors expressed as complex numbers. So, the meaning they use refers to these vector calculations involving complex numbers, and not the simple vectors I have been describing above.

John Manson explained this in an email to me. He said that Steinmetz (1865-1923) called the exponential form of a complex number a "vector". (Steinmetz was one of the foremost Electrical Engineers of his time). Steinmetz felt the exponential form, in the *Polar* notation, $A\,/\underline{\alpha}$, was the most simple and best way to deal with multiplication and division calculations of complex numbers.

There are other references to the use of this definition of "vector" by EE's. Such as Puchstein's May 12, 1921 Patent application that was titled *"Device For Making Vector Calculations"*. And, Weinbach refers to the *"Vector Slide Rule"* in his 1928 article. The name *"Log Log Vector Slide Rule"* is what he later recommended to K&E that they use for the K&E 4093. There is no question, when reading Puchstein's and Weinbach's comments that they are referring to the EE definition of "vectors" that specifically involve complex number calculations.

We also find this EE usage in the Pickett and Dietzgen manuals. Hartung in the Pickett manual uses the expression "...by ordinary vector methods..." in telling the user how to convert a complex component hyperbolic sinh, etc. to polar form. The Dietzgen N1725 manual uses the phrase "...the complex number can be converted to polar form in the usual manner" to describe the use of the vector form of conversion. The manual of the Hemmi 153 slide rule says, *"It calculates the complicated complex numbers, vector functions, circuit calculations, &c, &c, with ease and rapidity".*

The combination of hyperbolic scales with the decimally divided trigonometric scales on Weinbach's slide rule design allowed Engineers to expand their work into the complex number domain. For the first time calculations could be made directly for hyperbolic complex functions in both the *Polar and Cartesian* forms. This is why the introduction of a slide rule with hyperbolic scales in 1929 excited many Engineers - who up to that time had been making these extended type of hyperbolic complex number calculations by hand using published tables.

I guess we have answered Marion's question raised at the beginning of this section: "*What vendors had in mind when they used "vector" to name a slide rule?"* However, we have not answered the real mystery. Where did this definition of a "Vector" slide rule originate? This description did not come, it seems, from Puchstein's 1921 Patent application. Nor, from Weinbach in his 1928 writings. Maybe these two sowed the seed - but who was it to first write the above definition of a Vector Slide Rule *"as a slide rule with hyperbolic function scales"*?

What are Some of the Rarest Hyperbolic Slide Rules?

In the following list are my personal ideas of what I think are some of the rarest of the slide rules with hyperbolic scales. The main source used for compiling this is the eBay auctions in the USA. My introduction to eBay started in 1999, and literally thousands of slide rules of all types and descriptions have appeared in the auctions since then. During this time I have followed the listings quite closely. So, these rare slide rules are those that never have, or almost never have, appeared in these auctions. They are not picked because they are the most expensive, but because they are seldom seen and seem to exist in very small numbers. Slide rules such as the Aristo HyperLog 0972, and the F-C Mathemas 2/84 and 2/84N, appear on eBay often enough that I have purposely left them off the list. They are expensive to buy, but not really as rare as those listed.

I am sure many Readers will take exceptions as to what is shown, and even have strong opinions about other hyperbolic function slide rules that they think should be on the list. I really would like to hear from these Readers as to their other choices; and, maybe add these names to this list. In the interim here are my picks for the rare ones:

<u>First are the 10 inch or 25 cm. ones:</u>
Beacon (Korea) No. 315
Archimedes (Brazil) No. 21-D Vectolog
Eckel (USA) Engineer's Log Log (Circular Slide Rule)
Flying Fish (China) No. 6006
Graphoplex (France) Neperlog Hyperbolic No. 691a
Lutz 300 B (Japan/ USA)
Nikkei (Japan) 520 Duplex
Patrick (USA) Mark IV data log
Pickett & Eckel (USA) Model X-4 Executive
Relay (Japan) No. De 1008
Ricoh (Japan) No. 159
U.S. Blueprint (USA/Hemmi) No. 1893
<u>Next are the 20 inch or 50 cm. ones:</u>
Hemmi (Japan) Duplex Slide Rule No. 154
Hemmi (Japan) Duplex Slide Rule No. 275 Versions 1& 2
Hemmi (Japan) Duplex Slide Rule No. 275D
Post (USA/Hemmi) No. 1460

<u>About the list of Slide Rules with Hyperbolic Scales</u>

A separate listing, on this web site, shows the various hyperbolic slide rules that are known exist at the date of this writing. To see the list click on "Back" in the upper left hand corner of this page and the click on the List on the upper left corner of the web site introduction page. The list is alphabetically by manufacturer and gives a number of details about each of the hyperbolic slide rules they produced. This list begins in 1929 with the K&E 4093 and ends somewhere in the late 1970's, or early 1980's. The exact ending date of production and who the last manufacturer was will probably never be known.

My original listing of known hyperbolic slide rules was completed in March 2005. Since that time new additions to the list have been found. The updated current list shows an increase to 38 manufacturers and 128 slide rules. It was found that some of these manufacturers, particularly the Chinese, produced the same slide rule with similar model numbers, but used different names. So, if we eliminate these kinds of duplicates, I estimate we end up with somewhere around 30 different Companies – and probably about 100 different hyperbolic slide rules. These numbers, I'm afraid, are not too exact. It was just not possible, with the information available to date, to sort out all of the Chinese slide rules and the maker's names. And, we know these numbers are going to change in the future as new slide rules are reported.

On my original March 2005 list the slide rules were numbered from 1 through 109. The list is alphabetical by manufacturer. When revising the list I did not want to change the numbers each time a new slide rule was discovered and added somewhere on the list. So, when new rules were added they were inserted in place on the list under the manufacturer's name. The nearest old number was kept, but letter designations such as "a", "b", "c", etc. were added after the old number. This way each slide rule on the list can always be identified by its own assigned number and a letter designation where such has been added.

For each hyperbolic slide rule on the listing, four major items are shown: These are: (1) name and number; (2) arrangement of the scales on the rule; (3) gauge marks on the scales, if any; and (4) gauge marks on the cursor, if any. Not all slide rules have gauge marks on the scales and/or cursor. But these features are included in the listing because gauge marks add significant expanded calculating power to a slide rule.

These four items are the only variables that separate the slide rules included in the listing. So, there was no attempt, when all of the scales and gauge marks were the same, to include the variants of differences in a logo design, or name location, or cursor styles, etc.

This current list is a continuous working draft of what I hope in time will become a fairly thorough and accurate record of slide rules with hyperbolic scales. To complete it, I am asking for help from the International Slide Rule Group (ISRG) members, the Oughtred Society (OS) members, and other slide rule lovers to fill in the missing pieces. First, please look at the list for errors, and let me know what changes should be made. Second, let me know the name and details for any slide rule that has been omitted and should be added. I will be most happy to acknowledge the contributions that anyone makes to improve the listing. All errors in the list are mine alone.

The final goal is to include every known slide rule with hyperbolic scales for each manufacturer. In listing the rules that have the same name and number I'm looking for variations of the placements of the hyperbolic scales or other scales on the rule. Also, I'm trying to include variations in the gauge marks on the scales, and the marks on the cursor.

I have tried to include only actual details which have been verified. Where details are not complete the list is noted. If you see corrections to be made or feel something else should be added to a slide rule's description, please let me know. Front Scales denote the side of the slide rule that shows its name or number. This is not necessarily the side containing the hyperbolic scales. Red color denotes scales that are red color on the slide rule. Where red is missing in some of the listings it is because I do not own the rule or do not have a picture that shows the color of the scales. For some rules I do not have pictures that show all of the gauge marks on the scales and cursor. Please see where this is mentioned as I'm looking for help to add these marks. Also, far into the future, I am hoping to add the front and back pictures for all of the rules I own. I welcome all pictures of the rules that readers may send to me.

I wish to acknowledge the contributions made by Michael O'Leary, Pierre Vander Meulen, Richard Smith Hughes, John Fahey - and many others not mentioned here. Michael sent me emails with large size pictures of the variations of both K&E and Pickett rules that he had. Pierre did the same for many of the different Chinese slide rules that he

had purchased when he worked in China. Richard helped me further with the listing and details of many additional Chinese slide rules. John mailed me pictures of the Eckel circular slide rule.

These, and many other sources, were all important to recording an accurate listing for these rules. At the end of the listing all of the sources of information are shown in order to give each person who helped the proper credit. In the listing these sources are identified by initials at the far right for each rule. Refer to the footnotes at the end of the list for further historical details on some of the manufacturers and rules.

If you furnish additional information or corrections would you please give your source? Also, if you can send scanned pictures and/or a copy of the manual for these it would be most appreciated. Thanks for any help you may give.

Best regards, Bill Robinson (Email: wrobinson62@cox.net)

Text of 6/9/2008